The Pricing of First Generation Exotics

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Abstract

After vanilla options the first generation exotics account for the majority of the turnover of foreign exchange options. These are essentially digital options and all kind of barrier options such as single and double barrier options, single and double one-touch and no-touch options and corridor options. Here we present a formula catalogue for computing the theoretical value (TV) of such options in the Black-Scholes model.

1 Introduction

The pricing and hedging of the first generation exotic options in the Black–Scholes model is well understood. One takes a geometric Brownian motion with a risk-neutral drift and computes the discounted expected value of the respective option payoffs. This requires usually merely knowing a suitable probability density function, such as the joint density for the final time value and the maximum of a Brownian motion with drift. Computing the expectation results in the so-called *theoretical value (TV)* of the option. Although hardly any exotics trade at their theoretical value, this quantity is still a widely used reference value, and the market price of exotics is often computed as a sum of the theoretical value and a (possibly negative) adjustment. Here we outline the valuation of some of the most commonly used exotics: single barrier options in Section 2, digital options in Section 3, one-touch options in Section 4, double no-touch options in Section 5, corridors in Section 6, double barrier options in Section 7, fade-in-out options in Section 8, and slide-in corridors in Section 9. We will also illustrate how to handle different dates for the valuation of an option and the premium payment as well as different dates for the maturity and the delivery.

2 Single barrier options

In the model

$$dS_t = S_t[(r_d - r_f)dt + \sigma dW_t] \tag{1}$$

we consider the payoff for single barrier knock-out options

$$[\phi(S_T - K)]^+ I\!I_{\{\eta S_t > \eta B, 0 \le t \le T\}}, \tag{2}$$

where as usual the binary variable ϕ takes the values $+1$ for a call and -1 for a put, the binary variable η takes the values $+1$ if the barrier B is approached from above (down-and-out) and -1 if the barrier is approached from below (up-and-out). The strike is denoted by K and the maturity by T. Current time is denoted by t. The domestic and foreign interest rates are denoted by r_d and r_f respectively and the volatility by σ. To price knock-in options paying

$$[\phi(S_T - K)]^+ I\!I_{\{\min[\eta S_t] < \eta B\}} \tag{3}$$

we use the fact that

$$\text{kick-in} + \text{knock-out} = \text{vanilla}. \tag{4}$$

We denote the current value of the spot S_t by x and use the abbreviations listed in Table 1.

$X \triangleq \frac{\log(\frac{x}{K})+p}{\sigma\sqrt{\tau}}$	$\tau \triangleq T-t$
$\theta_{\pm} \triangleq \frac{r_d-r_f}{\sigma} \pm \frac{\sigma}{2}$	$x_1 \triangleq \frac{\log(\frac{x}{B})+p}{\sigma\sqrt{\tau}}$
$\mu \triangleq \sigma\theta_{-}$	$y \triangleq \frac{\log(\frac{B^2}{xK})+p}{\sigma\sqrt{\tau}}$
$n(t) \triangleq \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$	$y_1 \triangleq \frac{\log(\frac{B}{x})+p}{\sigma\sqrt{\tau}}$
$\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\,dt$	$w \triangleq \frac{\log(\frac{B}{x})+m\sigma^2\tau}{\sigma\sqrt{\tau}}$
$\lambda \triangleq 1+\frac{\mu}{\sigma^2}$	$z \triangleq 1-\frac{K}{B}$
$a \triangleq \frac{\mu}{\sigma^2}$	$d \triangleq e^{-r_d\tau}$
$p \triangleq (\mu+\sigma^2)\tau$	$f \triangleq e^{-r_f\tau}$

Table 1: Abbreviations used for the pricing formulas of single barrier options

2.1 Value

Computing the value of a barrier option in the Black-Scholes model boils down to knowing the joint density $f(x,y)$ for a Brownian motion with drift and its running extremum ($\eta=+1$ for a maximum and $\eta=-1$ for a minimum),

$$\left(W(T)+\theta_{-}T, \eta \min_{0\leq t\leq T}[\eta(W(t)+\theta_{-}t)]\right), \tag{5}$$

which is derived, e.g., in [4], and can be written as

$$f(x,y) = -\eta e^{\theta_{-}x-\frac{1}{2}\theta_{-}^2 T}\frac{2(2y-x)}{T\sqrt{2\pi T}}\exp\left\{-\frac{(2y-x)^2}{2T}\right\}, \quad \eta y \leq \min(0,\eta x). \tag{6}$$

Using the density (6) the value of a barrier option can be written as the following integral

$$\text{barrier}(S_0,\sigma,r_d,r_f,K,B,T) = e^{-r_dT}\mathbb{E}\left[[\phi(S_T-K)]^+ \mathbb{I}_{\{\eta S_t>\eta B, 0\leq t\leq T\}}\right] \tag{7}$$

$$= e^{-r_dT}\int_{x=-\infty}^{x=-\infty}\int_{\eta y\leq\min(0,\eta x)}[\phi(S_0e^{\sigma x}-K)]^+ \mathbb{I}_{\{\eta y>\eta\frac{1}{\sigma}\log\frac{B}{S_0}\}}f(x,y)\,dy\,dx. \tag{8}$$

Further details on how to evaluate this integral can be found in [4]. It produces four terms. We provide the four terms and summarize in Table 2 how they are used to find the value function.

option type	ϕ	η	in/out	reverse	combination
standard up-and-in call	$+1$	-1	-1	$K > B$	A_1
reverse up-and-in call	$+1$	-1	-1	$K \leq B$	$A_2 - A_3 + A_4$
reverse up-and-in put	-1	-1	-1	$K > B$	$A_1 - A_2 + A_4$
standard up-and-in put	-1	-1	-1	$K \leq B$	A_3
standard down-and-in call	$+1$	$+1$	-1	$K > B$	A_3
reverse down-and-in call	$+1$	$+1$	-1	$K \leq B$	$A_1 - A_2 + A_4$
reverse down-and-in put	-1	$+1$	-1	$K > B$	$A_2 - A_3 + A_4$
standard down-and-in put	-1	$+1$	-1	$K \leq B$	A_1
standard up-and-out call	$+1$	-1	$+1$	$K > B$	0
reverse up-and-out call	$+1$	-1	$+1$	$K \leq B$	$A_1 - A_2 + A_3 - A_4$
reverse up-and-out put	-1	-1	$+1$	$K > B$	$A_2 - A_4$
standard up-and-out put	-1	-1	$+1$	$K \leq B$	$A_1 - A_3$
standard down-and-out call	$+1$	$+1$	$+1$	$K > B$	$A_1 - A_3$
reverse down-and-out call	$+1$	$+1$	$+1$	$K \leq B$	$A_2 - A_4$
reverse down-and-out put	-1	$+1$	$+1$	$K > B$	$A_1 - A_2 + A_3 - A_4$
standard down-and-out put	-1	$+1$	$+1$	$K \leq B$	0

Table 2: The summands for the value of single barrier options

$$A_1 = \phi x f \mathcal{N}(\phi X) - \phi K d \mathcal{N}(\phi(X - \sigma\sqrt{\tau})) \tag{9}$$

$$A_2 = \phi x f \mathcal{N}(\phi x_1) - \phi K d \mathcal{N}(\phi(x_1 - \sigma\sqrt{\tau})) \tag{10}$$

$$A_3 = \phi \left(\frac{B}{x}\right)^{2\lambda-2} \left[x f \left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y) - K d \mathcal{N}(\eta(y - \sigma\sqrt{\tau})) \right] \tag{11}$$

$$\begin{aligned} A_4 = {} & \phi \frac{-2\mu}{\sigma^2 x} \left(\frac{B}{x}\right)^{2\lambda-2} \left[x f \left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y_1) - K d \mathcal{N}(\eta(y_1 - \sigma\sqrt{\tau})) \right] \\ & - \phi \left(\frac{B}{x}\right)^{2\lambda} f \mathcal{N}(\eta y_1) - \phi \eta f \left(\frac{B}{x}\right)^{2\lambda} n(y_1) z / \sigma\sqrt{\tau} \end{aligned} \tag{12}$$

2.2 Greeks

2.2.1 Delta

$$
\begin{aligned}
A_1 &= \phi f \mathcal{N}(\phi X) && (13)\\
A_2 &= \phi f \mathcal{N}(\phi x_1) + f n(x_1) z/\sigma\sqrt{\tau} && (14)\\
A_3 &= \phi \frac{-2\mu}{\sigma^2 x}\left(\frac{B}{x}\right)^{2\lambda-2}\left[x f\left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y) - K d\mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right]\\
&\quad -\phi\left(\frac{B}{x}\right)^{2\lambda} f\mathcal{N}(\eta y) && (15)\\
A_4 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[x f\left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y_1) - K d\mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right] && (16)
\end{aligned}
$$

2.2.2 Gamma

$$
\begin{aligned}
A_1 &= f n(X)/(x\sigma\sqrt{\tau}) && (17)\\
A_2 &= f n(x_1)/(x\sigma\sqrt{\tau})(1 - z x_1/\sigma\sqrt{\tau}) && (18)\\
C_3 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[x f\left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y) - K d\mathcal{N}(\eta(y-\sigma\sqrt{\tau}))\right] && (19)\\
B_3 &= \frac{-2\mu}{\sigma^2 x} C_3 - \phi\left(\frac{B}{x}\right)^{2\lambda} f\mathcal{N}(\eta y) && (20)\\
A_3 &= \frac{2\mu}{\sigma^2 x}(C_3/x - B_3) + \phi f B^{2\lambda}/x^{2\lambda+1}\left[2\lambda\mathcal{N}(\eta y) + \eta n(y)/\sigma\sqrt{\tau}\right] && (21)\\
C_4 &= \phi\left(\frac{B}{x}\right)^{2\lambda-2}\left[x f\left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y_1) - K d\mathcal{N}(\eta(y_1-\sigma\sqrt{\tau}))\right] && (22)\\
B_4 &= \frac{-2\mu}{\sigma^2 x} C_4 - \phi\left(\frac{B}{x}\right)^{2\lambda} f\mathcal{N}(\eta y_1) - \phi\eta f\left(\frac{B}{x}\right)^{2\lambda} n(y_1) z/\sigma\sqrt{\tau} && (23)\\
A_4 &= \frac{2\mu}{\sigma^2 x}(C_4/x - B_4) + \phi f B^{2\lambda}/x^{2\lambda+1}\left[2\lambda\mathcal{N}(\eta y_1) + \eta n(y_1)/\sigma\sqrt{\tau}\right]\\
&\quad +\phi\eta f z n(y_1)\left(\frac{B}{x}\right)^{2\lambda}/(x\sigma\sqrt{\tau})(2\lambda - y_1/\sigma\sqrt{\tau}) && (24)
\end{aligned}
$$

2.2.3 Theta

$$
\begin{aligned}
A_1 &= -\frac{1}{2}\sigma x f n(X)/\sqrt{\tau} + \phi x f \mathcal{N}(\phi X) r_f - \phi K d \mathcal{N}(\phi(X - \sigma\sqrt{\tau})) r_d && (25)\\
A_2 &= -\frac{1}{2}\sigma x f n(x_1) K/(B\sqrt{\tau}) + \phi x f \mathcal{N}(\phi x_1) r_f - \phi K d \mathcal{N}(\phi(x_1 - \sigma\sqrt{\tau})) r_d \\
&\quad - x f n(x_1) z y_1/(2\tau) && (26)\\
A_3 &= -\phi \left(\frac{B}{x}\right)^{2\lambda} x f \eta n(y) \frac{1}{2}\sigma/\sqrt{\tau} \\
&\quad + \phi \left(\frac{B}{x}\right)^{2\lambda-2} \left[r_f x f \left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y) - r_d K d \mathcal{N}(\eta(y - \sigma\sqrt{\tau}))\right] && (27)\\
A_4 &= -\phi \left(\frac{B}{x}\right)^{2\lambda} x f \eta n(y_1) \left[x_1/(2\tau) z + \frac{1}{2}\sigma K/(\sqrt{\tau} B)\right] \\
&\quad + \phi \left(\frac{B}{x}\right)^{2\lambda-2} \left[r_f x f \left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y_1) - r_d K d \mathcal{N}(\eta(y_1 - \sigma\sqrt{\tau}))\right] && (28)
\end{aligned}
$$

2.2.4 Vega

$$
\begin{aligned}
A_1 &= x f n(X)\sqrt{\tau} && (29)\\
A_2 &= x f n(x_1)(\sqrt{\tau} - x_1 z/\sigma) && (30)\\
B_3 &= \phi \left(\frac{B}{x}\right)^{2\lambda-2} \left[x f \left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y) - K d \mathcal{N}(\eta(y - \sigma\sqrt{\tau}))\right] && (31)\\
A_3 &= \frac{-4}{\sigma^3} \log\left(\frac{B}{x}\right)(r_d - r_f) B_3 + \phi \left(\frac{B}{x}\right)^{2\lambda} x f \eta n(y)\sqrt{\tau} && (32)\\
B_4 &= \phi \left(\frac{B}{x}\right)^{2\lambda-2} \left[x f \left(\frac{B}{x}\right)^2 \mathcal{N}(\eta y_1) - K d \mathcal{N}(\eta(y_1 - \sigma\sqrt{\tau}))\right] && (33)\\
A_4 &= \frac{-4}{\sigma^3} \log\left(\frac{B}{x}\right)(r_d - r_f) B_4 \\
&\quad + \phi \left(\frac{B}{x}\right)^{2\lambda} x f \eta n(y_1) \left[(\sqrt{\tau} - y_1/\sigma) z + \frac{K}{B}\sqrt{\tau}\right] && (34)
\end{aligned}
$$

2.3 Description via partial differential equation

We can describe a barrier option's value function also as a solution to a partial differential equation setup. Let $v(t,x)$ denote the value of the option at time t when the underlying is at x. Then $v(t,x)$ is the solution of

$$
\begin{aligned}
v_t + (r_d - r_f) x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} - r_d v &= 0,\ t \in [0,T],\ \eta x \geq \eta B, && (35)\\
v(T,x) &= [\phi(x - K)]^+,\ \eta x \geq \eta B, && (36)\\
v(t,B) &= 0,\ t \in [0,T]. && (37)
\end{aligned}
$$

2.4 Exponential barriers

We generalize the payoff to

$$[\phi(S_T - K)]^+ I\!I_{\{\eta S_t > \eta B(t), 0 \le t \le T\}}, \tag{38}$$

where the time-dependent barrier takes the exponential form

$$B(t) = Be^{\gamma t}. \tag{39}$$

The value function for the option with exponential barrier can be written as

$$\begin{aligned}
& \text{ExpBarrier}(S_0, \sigma, r_d, r_f, K, B, \gamma, T) \\
= \;& e^{-r_d T} I\!E\left[[\phi(S_T - K)]^+ I\!I_{\{\eta S_t > \eta B(t), 0 \le t \le T\}}\right] \\
= \;& e^{-r_d T} I\!E\left[[\phi(S_0 e^{\sigma W_T + \mu T} - K)]^+ I\!I_{\{\eta S_0 e^{\sigma W_t + \mu t} > \eta B e^{\gamma t}, 0 \le t \le T\}}\right] \\
= \;& e^{-(r_d - \gamma)T} I\!E\left[[\phi(S_0 e^{\sigma W_T + (\mu - \gamma)T} - K e^{-\gamma T})]^+ I\!I_{\{\eta S_0 e^{\sigma W_t + (\mu - \gamma)t} > \eta B, 0 \le t \le T\}}\right] \\
= \;& \text{barrier}(S_0, \sigma, r_d - \gamma, r_f, K e^{-\gamma T}, B, T).
\end{aligned} \tag{40}$$

Derivatives with respect to S_0, σ, r_d, r_f, B can instantly be taken from the function barrier, even higher order derivatives. Computing theta requires some caution, because T enters the function barrier also in the discounting factor for the strike.

3 Digital options

Digital options have a payoff

$$v(T) = I\!I_{\{\phi S_T \ge \phi K\}} \text{ cash-or-nothing}, \tag{41}$$

$$w(T) = S_T I\!I_{\{\phi S_T \ge \phi K\}} \text{ asset-or-nothing}. \tag{42}$$

In the cash-or-nothing case the payment of the fixed amount is in domestic currency, whereas in the asset-or-nothing case the payment is in foreign currency. We use the abbreviations

$$F \triangleq I\!E[S_T | S_t = x] = x e^{(r_d - r_f)\tau} \text{ (forward price of the underlying)}, \tag{43}$$

$$d_\pm \triangleq \frac{\ln \frac{x}{K} + \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}} = \frac{\ln \frac{F}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}, \tag{44}$$

$$\tilde{d}_\pm \triangleq \frac{\ln \frac{x}{K} - \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}}, \tag{45}$$

and obtain for the value functions

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = e^{-r_d \tau} \mathcal{N}(\phi d_-), \tag{46}$$

$$w(x, K, T, t, \sigma, r_d, r_f, \phi) = x e^{-r_f \tau} \mathcal{N}(\phi d_+). \tag{47}$$

3.1 Greeks

(Spot) Delta

$$\frac{\partial v}{\partial x} = \phi e^{-r_d\tau}\frac{n(d_-)}{x\sigma\sqrt{\tau}} \tag{48}$$

$$\frac{\partial w}{\partial x} = \phi e^{-r_f\tau}\frac{n(d_+)}{\sigma\sqrt{\tau}} + e^{-r_f\tau}\mathcal{N}(\phi d_+) \tag{49}$$

Gamma

$$\frac{\partial^2 v}{\partial x^2} = -\phi e^{-r_d\tau}\frac{n(d_-)d_+}{x^2\sigma^2\tau} \tag{50}$$

$$\frac{\partial^2 w}{\partial x^2} = -\phi e^{-r_f\tau}\frac{n(d_+)d_-}{x\sigma^2\tau} \tag{51}$$

Theta

$$\frac{\partial v}{\partial t} = e^{-r_d\tau}\left(r_d\mathcal{N}(\phi d_-) + \frac{\phi n(d_-)\tilde{d}_-}{2\tau}\right) \tag{52}$$

$$\frac{\partial w}{\partial t} = xe^{-r_f\tau}\left(r_f\mathcal{N}(\phi d_+) + \frac{\phi n(d_+)\tilde{d}_+}{2\tau}\right) \tag{53}$$

Vega

$$\frac{\partial v}{\partial \sigma} = -\phi e^{-r_d\tau}n(d_-)\frac{d_+}{\sigma} \tag{54}$$

$$\frac{\partial w}{\partial \sigma} = -\phi xe^{-r_f\tau}n(d_+)\frac{d_-}{\sigma} \tag{55}$$

Volga

$$\frac{\partial^2 v}{\partial \sigma^2} = -\phi e^{-r_d\tau}\frac{n(d_-)}{\sigma^2}(d_-d_+^2 - d_- - d_+) \tag{56}$$

$$\frac{\partial^2 w}{\partial \sigma^2} = -\phi xe^{-r_f\tau}\frac{n(d_+)}{\sigma^2}(d_+d_-^2 - d_+ - d_-) \tag{57}$$

Rho

$$\frac{\partial v}{\partial r_d} = e^{-r_d\tau}\left(-\tau\mathcal{N}(\phi d_-) + \frac{\phi\sqrt{\tau}n(d_-)}{\sigma}\right) \tag{58}$$

$$\frac{\partial v}{\partial r_f} = e^{-r_d\tau}\left(-\frac{\phi\sqrt{\tau}n(d_-)}{\sigma}\right) \tag{59}$$

$$\frac{\partial w}{\partial r_d} = xe^{-r_f\tau}\left(\frac{\phi\sqrt{\tau}n(d_+)}{\sigma}\right) \tag{60}$$

$$\frac{\partial w}{\partial r_f} = -xe^{-r_f\tau}\left(\tau\mathcal{N}(\phi d_+) + \frac{\phi\sqrt{\tau}n(d_+)}{\sigma}\right) \tag{61}$$

Dual Delta

$$\frac{\partial v}{\partial K} = -e^{-r_d\tau}\frac{\phi n(d_-)}{K\sigma\sqrt{\tau}} \tag{62}$$

$$\frac{\partial w}{\partial K} = -e^{-r_d\tau}\frac{\phi n(d_-)}{\sigma\sqrt{\tau}} \tag{63}$$

Dual Gamma

$$\frac{\partial^2 v}{\partial K^2} = \phi e^{-r_d\tau}\frac{n(d_-)}{K^2\sigma^2\tau}(\sigma\sqrt{\tau}-d_-) \tag{64}$$

$$\frac{\partial^2 w}{\partial K^2} = -\phi e^{-r_d\tau}\frac{n(d_-)d_-}{K\sigma^2\tau} \tag{65}$$

Dual Theta

$$\frac{\partial v}{\partial T} = -v_t \tag{66}$$

3.2 Foreign-domestic symmetry

One can directly verify the relationship

$$\frac{1}{x}v(x,K,T,t,\sigma,r_d,r_f,\phi) = w(\frac{1}{x},\frac{1}{K},T,t,\sigma,r_f,r_d,-\phi). \tag{67}$$

The reason is that the value of an option can be computed both in a domestic as well as in a foreign scenario. We consider the example of S_t modeling the exchange rate of EUR/USD. In New York, the cash-or-nothing digital call option costs $v(x,K,T,t,\sigma,r_{usd},r_{eur},1)$ USD and hence $v(x,K,T,t,\sigma,r_{usd},r_{eur},1)/x$ EUR. If it ends in the money, the holder receives 1 USD. For a Frankfurt-based holder of the same option, receiving one USD means receiving asset-or-nothing, where he uses reciprocal values for spot and strike and for him domestic currency is the one that's foreign to the New Yorker and vice versa. Since S_t and $\frac{1}{S_t}$ have the same volatility, the New York value and the Frankfurt value must agree, which leads to (67).

3.3 Relationship between cash, asset and vanilla

The simple equation of payoffs

$$\phi(w(T)-Kv(T)) = [\phi(S_T-K)]^+ \tag{68}$$

leads to the formula

$$\begin{aligned} &\text{vanilla}(x,K,T,t,\sigma,r_d,r_f,\phi) \\ =\;& \phi[w(x,K,T,t,\sigma,r_d,r_f,\phi) - Kv(x,K,T,t,\sigma,r_d,r_f,\phi)]. \end{aligned} \tag{69}$$

3.4 Static hedge using vertical spreads

The mathematical derivative of the positive part function

$$\mathbb{I}_{\{\phi S_t \geq \phi K\}} = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\left[(\phi(S_T-(K-\phi\epsilon)))^+ - (\phi(S_T-(K+\phi\epsilon)))^+\right] \tag{70}$$

leads to an approximate static hedge (and hence price)

$$v(x,K,T,t,\sigma,r_d,r_f,\phi) \approx \tag{71}$$
$$\frac{1}{2\epsilon}\left[\text{vanilla}(x,K-\phi\epsilon,T,t,\sigma,r_d,r_f,\phi) - \text{vanilla}(x,K+\phi\epsilon,T,t,\sigma,r_d,r_f,\phi)\right]$$

for small $\epsilon > 0$. In practice, arbitrarily small ϵ corresponds to arbitrarily large nominal amounts of the vanilla options and can thus not be chosen arbitrarily small. Furthermore, there will be different volatilities for the bid and ask price of the vanilla options, which lead to a more realistic pricing for digital options using this approximation.

3.4.1 Greeks in the static hedge

Static hedges normally perform well hedging the actual model variable risk like delta, gamma and theta. In this static hedge even the model *parameter* uncertainty vega is hedged. The hedge vega is given by

$$\sqrt{\tau}xe^{-r_f\tau}\frac{n(d_+^{K-\phi\epsilon}) - n(d_+^{K+\phi\epsilon})}{2\epsilon}, \tag{72}$$

$$d_\pm^K \triangleq \frac{\ln\frac{F}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}. \tag{73}$$

Replacing the difference quotient by its derivative at K we obtain

$$\sqrt{\tau}xe^{-r_f\tau}\frac{n(d_+^{K-\phi\epsilon}) - n(d_+^{K+\phi\epsilon})}{2\epsilon} \tag{74}$$
$$\approx \phi\sqrt{\tau}xe^{-r_f\tau}\cdot n(d_+)d_+\frac{-1}{K\sigma\sqrt{\tau}} \tag{75}$$
$$= -\phi e^{-r_d\tau}n(d_-)\frac{d_+}{\sigma}, \tag{76}$$

which is the vega of the digital option.

3.5 Handling different dates for valuation, payment, expiry and delivery for digital options with two barriers

Generally pricing foreign exchange options requires handling different dates for *valuation*, *payment*, *expiry* and *delivery*. We denote these by t, $t+T_p$, T_e and T_d respectively. The valuation date t is also called the *horizon*.
Let us consider different interest rates for the respective time intervals, i.e., let

- $r_t^{T_p}$ be the interest rate between t and $t+T_p$,
- $r_t^{T_e}$ be the interest rate between t and $t+T_e$, where the spot is modeled by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW \tag{77}$$

  with $\mu_t \triangleq r_t^{T_e d} - r_t^{T_e f}$, and
- $r_t^{T_d}$ be the interest rate between t and $t+T_d$.

We consider a generalized payoff of the digital option by taking both a lower barrier L and an higher barrier H,

$$I\!I_{\{L\le S_{T_e}\le H\}}. \tag{78}$$

3.5.1 Pricing

On $[t, T_e]$, the price of the option is

$$v(t) = e^{r_t^{T_p} T_p} \mathbb{E}^t \left[e^{-r_t^{T_d}(T_d - t)} \mathbb{I}_{\{L \le S_{T_e} \le H\}} \right], \tag{79}$$

where

$$S_{T_e} = S_t e^{\left(\mu_t - \frac{1}{2}\sigma_t^2\right)(T_e - t) + \sigma_t W_t}, \tag{80}$$

And the symbol $\mathbb{E}^t$ means the expectation based on the information available up to time t. Computing the integral yields

$$v(t) = e^{r_t^{T_p} T_p - r_t^{T_d}(T_d - t)} \left(\mathcal{N}(d_L) - \mathcal{N}(d_H) \right) \tag{81}$$

with

$$d_L \triangleq \frac{1}{\sigma_t \sqrt{T_e - t}} \left(\ln \left(\frac{S}{L} \right) + \left(\mu_t + \frac{1}{2}\sigma_t^2 \right) (T_e - t) \right), \tag{82}$$

$$d_H \triangleq \frac{1}{\sigma_t \sqrt{T_e - t}} \left(\ln \left(\frac{S}{H} \right) + \left(\mu_t + \frac{1}{2}\sigma_t^2 \right) (T_e - t) \right). \tag{83}$$

On $[T_e, T_d]$ the price is

$$v(t) = e^{r_t^{T_p} T_p - r_d(T_d - t)} \mathbb{I}_{\{L \le S_{T_e} \le H\}}. \tag{84}$$

4 One-touch options

We consider now options paying

$$R \mathbb{I}_{\{\tau_B \le T\}}, \tag{85}$$

$$\tau_B \triangleq \inf\{t \ge 0 : \eta S_t \le \eta B\}. \tag{86}$$

This type of option pays a domestic cash amount R if a barrier B is hit any time before expiry. We use the binary variable η to describe whether B is a lower barrier ($\eta = 1$) or an upper barrier ($\eta = -1$). The stopping time τ_B is called the *first hitting time*. The option can be either viewed as the rebate portion of a knock-out barrier option or as an American cash-or-nothing digital option. It is also sometimes called *one-touch option*, *one-touch-digital* or *hit option*. The modified payoff $R\mathbb{I}_{\{\tau_B \ge T\}}$ describes a rebate which is being paid if a knock-in-option has not knocked in by the time it expires and can be valued similarly simply by exploiting the identity

$$R \mathbb{I}_{\{\tau_B \le T\}} + R \mathbb{I}_{\{\tau_B \ge T\}} = R. \tag{87}$$

We will further distinguish whether the rebate is paid at hit ($\omega = 0$) or at end ($\omega = 1$) and use the abbreviations

$$\vartheta_- \triangleq \sqrt{\theta_-^2 + 2(1 - \omega) r_d}, \tag{88}$$

$$e_\pm \triangleq \frac{\pm \log \frac{x}{B} - \sigma \vartheta_- \tau}{\sigma \sqrt{\tau}}. \tag{89}$$

4.1 Pricing

The value of the one-touch option turns out to be

$$v(t,x) = Re^{-\omega r_d \tau}\left[\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} \mathcal{N}(-\eta e_+) + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} \mathcal{N}(\eta e_-)\right]. \tag{90}$$

Note that $\vartheta_- = |\theta_-|$ for rebates paid at end ($\omega = 1$).

4.2 Greeks

4.2.1 Delta

$$\begin{aligned} v_x(t,x) = -\frac{Re^{-\omega r_d \tau}}{\sigma x} & \left\{\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} \left[(\theta_- + \vartheta_-)\mathcal{N}(-\eta e_+) + \frac{\eta}{\sqrt{\tau}} n(e_+)\right]\right. \\ & \left. + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} \left[(\theta_- - \vartheta_-)\mathcal{N}(\eta e_-) + \frac{\eta}{\sqrt{\tau}} n(e_-)\right]\right\} \end{aligned} \tag{91}$$

4.2.2 Theta

$$\begin{aligned} v_t(t,x) &= \omega r_d v(t,x) + \frac{\eta Re^{-\omega r_d \tau}}{2\tau}\left[\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} n(e_+) e_- - \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} n(e_-) e_+\right] \\ &= \omega r_d v(t,x) + \frac{\eta Re^{-\omega r_d \tau}}{\sigma \tau^{(3/2)}}\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} n(e_+) \log\left(\frac{B}{x}\right). \end{aligned} \tag{92}$$

The computation exploits the identities (109), (110) and (111) derived below.

4.2.3 Gamma

Gamma can be obtained using $v_{xx} = \frac{2}{\sigma^2 x^2}[r_d v - v_t - (r_d - r_f) x v_x]$ and turns out to be

$$\begin{aligned} v_{xx}(t,x) = & \frac{2Re^{-\omega r_d \tau}}{\sigma^2 x^2} \cdot \\ & \left\{\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} \mathcal{N}(-\eta e_+)\left[r_d(1-\omega) + (r_d - r_f)\frac{\theta_- + \vartheta_-}{\sigma}\right]\right. \\ & + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} \mathcal{N}(\eta e_-)\left[r_d(1-\omega) + (r_d - r_f)\frac{\theta_- - \vartheta_-}{\sigma}\right] \\ & + \eta\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} n(e_+)\left[-\frac{e_-}{\tau} + \frac{r_d - r_f}{\sigma\sqrt{\tau}}\right] \\ & \left. + \eta\left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} n(e_-)\left[\frac{e_+}{\tau} + \frac{r_d - r_f}{\sigma\sqrt{\tau}}\right]\right\}. \end{aligned} \tag{93}$$

4.2.4 Vega

To compute vega we use the identities

$$\frac{\partial\theta_-}{\partial\sigma} = -\frac{\theta_+}{\sigma}, \tag{94}$$

$$\frac{\partial\vartheta_-}{\partial\sigma} = -\frac{\theta_-\theta_+}{\sigma\vartheta_-}, \tag{95}$$

$$\frac{\partial e_\pm}{\partial\sigma} = \pm\frac{\log\frac{B}{x}}{\sigma^2\sqrt{\tau}} + \frac{\theta_-\theta_+}{\sigma\vartheta_-}\sqrt{\tau}, \tag{96}$$

$$A_\pm \stackrel{\Delta}{=} \frac{\partial}{\partial\sigma}\frac{\theta_- \pm \vartheta_-}{\sigma} = -\frac{1}{\sigma^2}\left[\theta_+ + \theta_- \pm \left(\frac{\theta_-\theta_+}{\vartheta_-} + \vartheta_-\right)\right], \tag{97}$$

and obtain

$$\begin{aligned} v_\sigma(t,x) = \; & Re^{-\omega r_d\tau}\cdot \\ & \left\{\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\left[\mathcal{N}(-\eta e_+)A_+\log\left(\frac{B}{x}\right) - \eta n(e_+)\frac{\partial e_+}{\partial\sigma}\right]\right. \\ & \left. + \left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\left[\mathcal{N}(\eta e_-)A_-\log\left(\frac{B}{x}\right) + \eta n(e_-)\frac{\partial e_-}{\partial\sigma}\right]\right\}. \end{aligned} \tag{98}$$

4.3 Knock-out probability

The risk-neutral probability of knocking out is given by

$$\mathbb{P}[\tau_B \le T] = \mathbb{E}\left[\mathbb{I}_{\{\tau_B \le T\}}\right] = \frac{1}{R}e^{r_d T}v(0, S_0). \tag{99}$$

4.4 Properties of the first hitting time τ_B

As derived, e.g., in [4], the first hitting time

$$\tilde\tau \stackrel{\Delta}{=} \inf\{t \ge 0 : \theta t + W(t) = x\} \tag{100}$$

of a Brownian motion with drift θ and hit level $x > 0$ has the density

$$\mathbb{P}[\tilde\tau \in dt] = \frac{x}{t\sqrt{2\pi t}}\exp\left\{-\frac{(x-\theta t)^2}{2t}\right\}dt, \quad t > 0, \tag{101}$$

the cumulative distribution function

$$\mathbb{P}[\tilde\tau \le t] = \mathcal{N}\left(\frac{\theta t - x}{\sqrt{t}}\right) + e^{2\theta x}\mathcal{N}\left(\frac{-\theta t - x}{\sqrt{t}}\right), \quad t > 0, \tag{102}$$

the Laplace-transform

$$\mathbb{E}e^{-\alpha\tilde\tau} = \exp\left\{x\theta - x\sqrt{2\alpha + \theta^2}\right\}, \quad \alpha > 0,\; x > 0, \tag{103}$$

and the property

$$\mathbb{P}[\tilde\tau < \infty] = \begin{cases} 1 & \text{if } \theta \ge 0 \\ e^{2\theta x} & \text{if } \theta < 0 \end{cases}. \tag{104}$$

For upper barriers $B > S_0$ we can now rewrite the first passage time τ_B as

$$\begin{aligned}\tau_B &= \inf\{t \geq 0 : S_t = B\} \\ &= \inf\left\{t \geq 0 : W_t + \theta_- t = \frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)\right\}. \end{aligned} \tag{105}$$

The density of τ_B is hence

$$\mathbb{P}[\tilde{\tau_B} \in dt] = \frac{\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right) - \theta_- t)^2}{2t}\right\}, \quad t > 0. \tag{106}$$

4.5 Derivation of the value function

Using the density (106) the value of the paid-at-end ($\omega = 1$) upper rebate ($\eta = -1$) option can be written as

$$\begin{aligned} v(T, S_0) &= Re^{-r_d T}\mathbb{E}\left[\mathbb{I}_{\{\tau_B \leq T\}}\right] \qquad (107) \\ &= Re^{-r_d T}\int_0^T \frac{\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right) - \theta_- t)^2}{2t}\right\} dt. \end{aligned}$$

To evaluate this integral, we introduce the notation

$$e_\pm(t) \triangleq \frac{\pm\log\frac{S_0}{B} - \sigma\theta_- t}{\sigma\sqrt{t}} \tag{108}$$

and list the properties

$$e_-(t) - e_+(t) = \frac{2}{\sqrt{t}}\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right), \tag{109}$$

$$n(e_+(t)) = \left(\frac{B}{S_0}\right)^{-\frac{2\theta_-}{\sigma}} n(e_-(t)), \tag{110}$$

$$\frac{\partial e_\pm(t)}{\partial t} = \frac{e_\mp(t)}{2t}. \tag{111}$$

We evaluate the integral in (107) by rewriting the integrand in such a way that the coefficients of the exponentials are the inner derivatives of the exponentials using properties (109),(110) and (111),

$$\begin{aligned} &\int_0^T \frac{\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(\frac{1}{\sigma}\log\left(\frac{B}{S_0}\right) - \theta_- t)^2}{2t}\right\} dt \\ &= \frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)\int_0^T \frac{1}{t^{(3/2)}} n(e_-(t))\,dt \\ &= \int_0^T \frac{1}{2t} n(e_-(t))[e_-(t) - e_+(t)]\,dt \\ &= -\int_0^T n(e_-(t))\frac{e_+(t)}{2t} + \left(\frac{B}{S_0}\right)^{\frac{2\theta_-}{\sigma}} n(e_+(t))\frac{e_-(t)}{2t}\,dt \\ &= \left(\frac{B}{S_0}\right)^{\frac{2\theta_-}{\sigma}} \mathcal{N}(e_+(T)) + \mathcal{N}(-e_-(T)). \end{aligned} \tag{112}$$

The computation for lower barriers ($\eta = 1$) is similar.

5 Double no-touch options

We use the notation of Section 3.5. A double no-touch option pays off

$$\mathbb{I}_{\{L\le\min_{[0,T_e]}S_t<\max_{[0,T_e]}S_t\le H\}} \tag{113}$$

5.1 Pricing

On $[t,\tau]$, the price of the option is

$$v(t)=e^{r_t^{T_p}T_p}\mathbb{E}^t\left[e^{-r_t^{T_d}(T_d-t)}\mathbb{I}_{\{L\le\min_{[0,T_e]}S_t<\max_{[0,T_e]}S_t\le H\}}\right], \tag{114}$$

on $[\tau,T_d]$,

$$v(t)=e^{r_t^{T_p}T_p-r_d(T_d-t)}\mathbb{I}_{\{L\le\min_{[0,T_e]}S_t<\max_{[0,T_e]}S_t\le H\}}. \tag{115}$$

To compute the expectation, let us introduce the stopping time

$$\tau\stackrel{\Delta}{=}\min\left\{\inf\left\{t\in[0,T_e]|S_t=L\text{ or }S_t=H\right\},T_e\right\} \tag{116}$$

and the notation

$$\tilde{\theta}\stackrel{\Delta}{=}\frac{r_d-r_f-\frac{1}{2}\sigma^2}{\sigma} \tag{117}$$

$$\tilde{h}\stackrel{\Delta}{=}\frac{1}{\sigma}\ln\frac{H}{S_t} \tag{118}$$

$$\tilde{l}\stackrel{\Delta}{=}\frac{1}{\sigma}\ln\frac{L}{S_t} \tag{119}$$

$$\theta\stackrel{\Delta}{=}\tilde{\theta}\sqrt{T_e-t} \tag{120}$$

$$h\stackrel{\Delta}{=}\tilde{h}/\sqrt{T_e-t} \tag{121}$$

$$l\stackrel{\Delta}{=}\tilde{l}/\sqrt{T_e-t} \tag{122}$$

$$y_n\stackrel{\Delta}{=}2n(h-l)-\theta \tag{123}$$

$$n_T(x)\stackrel{\Delta}{=}\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2T}\right). \tag{124}$$

The joint distribution of the maximum and the minimum of a Brownian motion can be taken from [1] and is given by

$$\mathbb{P}\left[\tilde{l}\le\min_{[0,T]}W_t<\max_{[0,T]}W_t\le\tilde{h}\right]=\int_{\tilde{l}}^{\tilde{h}}k_T(x)\,dx \tag{125}$$

with

$$k_T(x)=\sum_{n=-\infty}^{\infty}\left[n_T(x+2n(\tilde{h}-\tilde{l}))-n_T(x-2\tilde{h}+2n(\tilde{h}-\tilde{l}))\right]. \tag{126}$$

Hence the joint density of the maximum and the minimum of a Brownian motion with drift $\tilde{\theta}$, $W_t^{\tilde{\theta}}\stackrel{\Delta}{=}W_t+\tilde{\theta}t$, is given by

$$k_T^{\tilde{\theta}}(x)=k_T(x)\exp\left\{\tilde{\theta}x-\frac{1}{2}\tilde{\theta}^2T\right\}. \tag{127}$$

We obtain for the price of the option on $[t, \tau]$

$$\begin{aligned} v(t) &= e^{r_t^{T_p} T_p - r_d(T_d - t)} \mathbb{I}_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}} \\ &= e^{r_t^{T_p} T_p - r_d(T_d - t)} \mathbb{I}_{\{\tilde{l} \le \min_{[0,T_e]} W_t^{\tilde{\theta}} < \max_{[0,T_e]} W_t^{\tilde{\theta}} \le \tilde{h}\}} \\ &= e^{r_t^{T_p} T_p - r_d(T_d - t)} \int_{\tilde{l}}^{\tilde{h}} k_{(T_e - t)}^{\tilde{\theta}}(x) dx \\ &= e^{r_t^{T_p} T_p - r_t^{T_d}(T_d - t)} \\ &\quad \cdot \sum_{n=-\infty}^{\infty} \Big[e^{-2n\theta(h-l)} \left\{ \mathcal{N}(h + y_n) - \mathcal{N}(l + y_n) \right\} \\ &\quad - e^{-2n\theta(h-l) + 2\theta h} \left\{ \mathcal{N}(h - 2h + y_n) - \mathcal{N}(l - 2h + y_n) \right\} \Big] \end{aligned} \tag{128}$$

and on $[\tau, T_d]$

$$v(t) = e^{r_t^{T_p} T_p - r_d(T_d - t)} \mathbb{I}_{\{L \le \min_{[0,T_e]} S_t < \max_{[0,T_e]} S_t \le H\}}. \tag{129}$$

6 Corridors

6.1 Corridor of digital options

Let us consider N digital options on the same underlying S_t, with the same barrier levels L and H, and with the same delivery date T_d. Let us assume that the expiry dates T_e^i depend on the digital i.
The payoff of the corridor of these N digital options is

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{L \le S_{T_e^i} \le H\}}, \tag{130}$$

whence the price of the corridor is

$$v(t) = \frac{1}{N} e^{r_t^{T_p} T_p - r_t^{T_d}(T_d - t)} \left[\left(\mathcal{N}(d_L^i) - \mathcal{N}(d_H^i) \right) \mathbb{I}_{\{t < T_e^i\}} + \mathbb{I}_{\{L^i \le S_{T_e} \le H^i\}} \mathbb{I}_{\{t \ge T_e^i\}} \right] \tag{131}$$

with

$$d_L^i \triangleq \frac{1}{\sigma_t \sqrt{T_e^i - t}} \left[\ln\left(\frac{S_t}{L}\right) + \left(\mu_t - \frac{1}{2}\sigma_t^2\right)(T_e^i - t) \right], \tag{132}$$

$$d_H^i \triangleq \frac{1}{\sigma_t \sqrt{T_e^i - t}} \left[\ln\left(\frac{S_t}{H}\right) + \left(\mu_t - \frac{1}{2}\sigma_t^2\right)(T_e^i - t) \right]. \tag{133}$$

6.2 Corridor of no-touch options

Let us consider N no-touch options on the same underlying S_t with the same barrier levels L and H and with the same delivery date T_d. Let us assume that the expiry dates T_e^i depend on the no-touch option i.
The payoff of the corridor is

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{L \le \min_{[0,T_e^i]} S_t < \max_{[0,T_e^i]} S_t \le H\}}. \tag{134}$$

The price of the corridor is now a sum of prices of double-no-touch options with maturities T_e^i.

7 Double barrier options

A double barrier option is an option which pays off

$$(\phi\,(S_{T_e}-K))^+\,\mathbb{I}_{\{L<\min_{[0,T_e]} S_t \le \max_{[0,T_e]} S_t < H\}}, \tag{135}$$

where K denotes the strike and the notation is the same as in Section 3.5.

7.1 Pricing

The distribution of S_{T_e} conditioned on not having reached the upper barrier H and the lower barrier L is

$$\begin{aligned}
&e^{-\frac{1}{2}\lambda^2(T_e-t)+\frac{\lambda}{\sigma}\ln\frac{S_{T_e}}{S_t}}\times\\
&\sum_{n=-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\left[\exp\left(-\frac{1}{2\sigma^2(T_e-t)}\left(\ln\frac{S_{T_e}}{S_t}+2n\ln\frac{H}{L}\right)^2\right)\right.\\
&\left.-\exp\left(-\frac{1}{2\sigma^2(T_e-t)}\left(\ln\frac{H^2}{S_{T_e}S_t}+2n\ln\frac{H}{L}\right)^2\right)\right]\mathbb{I}_{\{L<S_{T_e}<H\}}
\end{aligned} \tag{136}$$

with

$$\lambda \triangleq \frac{\mu}{\sigma}-\frac{\sigma}{2}. \tag{137}$$

To price the option, let us introduce the stopping time

$$\tau \triangleq \min\{\inf\{t\in[0,T_e]|S_t=L \text{ or } S_t=H\},T_e\} \tag{138}$$

The price of the option on $[t,\tau]$ is

$$v(t)=e^{-r_{T_e}(T_e-t)}\mathbb{E}^t\left[(\phi\,(S_T-K))^+\,\mathbb{I}_{\{L<\min_{[t,T_e]} S_s \le \max_{[t,T_e]} S_s < H\}}\right] \tag{139}$$

and on $[\tau,T_d]$

$$v(t)=0. \tag{140}$$

First we consider the case of a call option ($\phi=1$), where we need to evaluate

$$\begin{aligned}
v(t) &= e^{-r_{T_e}(T_e-t)}\mathbb{E}^t\left[(S_{T_e}-K)^+\,\mathbb{I}_{\{L<\min_{[t,T_e]} S_s \le \max_{[t,T_e]} S_s < H\}}\right]\\
&= e^{-r_{T_e}(T_e-t)}\\
&\quad\times\mathbb{E}^t\left[(S_{T_e}-K)\,\mathbb{I}_{\{K<S_T<H\}}\mathbb{I}_{\{L<\min_{[t,T_e]} S_s \le \max_{[t,T_e]} S_s < H\}}\right]\\
&= e^{-r_{T_e}(T_e-t)}\int_K^H (S_{T_e}-K)\,e^{-\frac{1}{2}\lambda^2(T_e-t)+\frac{\lambda}{\sigma}\ln\frac{S_{T_e}}{S_t}}\times\\
&\quad\sum_{n=-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\left[\exp\left(-\frac{1}{2\sigma^2(T_e-t)}\left(\ln\frac{S_{T_e}}{S_t}+2n\ln\frac{H}{L}\right)^2\right)\right.\\
&\quad\left.-\exp\left(-\frac{1}{2\sigma^2(T_e-t)}\left(\ln\frac{H^2}{S_{T_e}S_t}+2n\ln\frac{H}{L}\right)^2\right)\right]d\mathbb{P}_{T_e}\\
&= e^{-r_{T_e}(T_e-t)}\sum_{n=-\infty}^{+\infty}(S_t(Q_1^n-Q_2^n)-K(P_1^n-P_2^n)),
\end{aligned} \tag{141}$$

using the notation

$$A_K \triangleq \frac{\ln \frac{L}{S_t}}{\sigma\sqrt{T_e - t}}, \tag{142}$$

$$A_H \triangleq \frac{\ln \frac{H}{S_t}}{\sigma\sqrt{T_e - t}}, \tag{143}$$

$$A_{LH} \triangleq \frac{\ln \frac{H}{L}}{\sigma\sqrt{T_e - t}}, \tag{144}$$

$$x \triangleq \frac{\ln \frac{S_{T_e}}{S_t}}{\sigma\sqrt{T_e - t}}, \tag{145}$$

and

$$\begin{aligned} Q_1^n &\triangleq \int_{A_K}^{A_H} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2(T_e-t)+2\lambda\sqrt{T_e-t}x-\frac{1}{2}(x+2nA_{LH})^2} dx \\ &= \int_{A_K}^{A_H} \frac{1}{\sqrt{2\pi}} e^{-4n\lambda A_{LH}\sqrt{T_e-t}+\frac{3}{2}\lambda^2(T_e-t)} e^{-\frac{1}{2}(x-(2\lambda\sqrt{T_e-t}-2nA_{LH}))^2} dx \\ &= e^{-\lambda\sqrt{T_e-t}(4nA_{LH})+\frac{3}{2}\lambda^2(T_e-t)} \\ &\times \left\{ \mathcal{N}\left(A_H - 2\lambda\sqrt{T_e-t} + 2nA_{LH}\right) \right. \\ &\left. -\mathcal{N}\left(A_K - 2\lambda\sqrt{T_e-t} + 2nA_{LH}\right) \right\}, \end{aligned} \tag{146}$$

$$\begin{aligned} Q_2^n &\triangleq \int_{A_K}^{A_H} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2(T_e-t)+2\lambda\sqrt{T_e-t}x-\frac{1}{2}(x-2A_H-2nA_{LH})^2} dx \\ &= e^{\lambda\sqrt{T_e-t}(4A_H+4nA_{LH})+\frac{3}{2}\lambda^2(T_e-t)} \\ &\times \left\{ \mathcal{N}\left(-(A_H + 2nA_{LH} + 2\lambda\sqrt{T_e-t})\right) \right. \\ &\left. -\mathcal{N}\left(A_K - (2\lambda\sqrt{T_e-t} + 2A_H + 2nA_{LH})\right) \right\}, \end{aligned} \tag{147}$$

$$\begin{aligned} P_1^n &\triangleq \int_{A_K}^{A_H} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2(T_e-t)+\lambda\sqrt{T_e-t}x-\frac{1}{2}(x+2nA_{LH})^2} dx \\ &= \int_{A_K}^{A_H} \frac{1}{\sqrt{2\pi}} e^{-2n\lambda A_{LH}\sqrt{T_e-t}} e^{-\frac{1}{2}(x-(\lambda\sqrt{T_e-t}-2nA_{LH}))^2} dx \\ &= e^{-\lambda\sqrt{T_e-t}(2nA_{LH})} \\ &\times \left\{ \mathcal{N}\left(A_H - \lambda\sqrt{T_e-t} + 2nA_{LH}\right) \right. \\ &\left. -\mathcal{N}\left(A_K - \lambda\sqrt{T_e-t} + 2nA_{LH}\right) \right\}, \end{aligned} \tag{148}$$

$$\begin{aligned} P_2^n &\triangleq \int_{A_K}^{A_H} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2(T_e-t)+\lambda\sqrt{T_e-t}x-\frac{1}{2}(x-2A_H-2nA_{LH})^2} dx \\ &= e^{\lambda\sqrt{T_e-t}(2A_H+2nA_{LH})} \\ &\times \left\{ \mathcal{N}\left(-(A_H + 2nA_{LH} + \lambda\sqrt{T_e-t})\right) \right. \\ &\left. -\mathcal{N}\left(A_K - (\lambda\sqrt{T_e-t} + 2A_H + 2nA_{LH})\right) \right\}. \end{aligned} \tag{149}$$

We obtain for the price of the call option on $[t, \tau]$

$$\begin{aligned} v(t) &= e^{-r_{T_e}(T_e-t)} \Bigg\{ S_t \Bigg[\sum_{n=-\infty}^{+\infty} e^{-\lambda\sqrt{T_e-t}(4nA_{LH})+\frac{3}{2}\lambda^2(T_e-t)} \\ &\times \Big\{ \mathcal{N}\Big(A_H - 2\lambda\sqrt{T_e-t} + 2nA_{LH}\Big) - \mathcal{N}\Big(A_K - 2\lambda\sqrt{T_e-t} + 2nA_{LH}\Big)\Big\} \\ &- \sum_{n=-\infty}^{+\infty} e^{\lambda\sqrt{T_e-t}(4A_H+4nA_{LH})+\frac{3}{2}\lambda^2(T_e-t)} \\ &\times \Big\{ \mathcal{N}\Big(-(A_H + 2nA_{LH} + 2\lambda\sqrt{T_e-t})\Big) - \mathcal{N}\Big(A_K - (2\lambda\sqrt{T_e-t} + 2A_H + 2nA_{LH})\Big)\Big\}\Bigg] \\ &- K \Bigg[\sum_{n=-\infty}^{+\infty} e^{-\lambda\sqrt{T_e-t}(2nA_{LH})} \\ &\times \Big\{ \mathcal{N}\Big(A_H - \lambda\sqrt{T_e-t} + 2nA_{LH}\Big) - \mathcal{N}\Big(A_K - \lambda\sqrt{T_e-t} + 2nA_{LH}\Big)\Big\} \\ &- \sum_{n=-\infty}^{+\infty} e^{\lambda\sqrt{T_e-t}(2A_H+2nA_{LH})} \\ &\times \Big\{ \mathcal{N}\Big(-(A_H + 2nA_{LH} + \lambda\sqrt{T_e-t})\Big) - \mathcal{N}\Big(A_K - (\lambda\sqrt{T_e-t} + 2A_H + 2nA_{LH})\Big)\Big\}\Bigg]\Bigg\} \end{aligned} \tag{150}$$

and zero otherwise.

Similarly we obtain for the price of the put ($\phi = 1$)

$$\begin{aligned} v(t) &= e^{-r_{T_e}(T_e-t)} \mathbb{E}^t \Big[(K - S_{T_e})^+ \mathbb{I}_{\{L < \min_{[t,T_e]} S_s \le \max_{[t,T_e]} S_s < H\}} \Big] \\ &= e^{-r_{T_e}(T_e-t)} \Bigg\{ K \Bigg[\sum_{n=-\infty}^{+\infty} e^{-\lambda\sqrt{T_e-t}(2nA_{LH})} \\ &\times \Big\{ \mathcal{N}\Big(A_K - \lambda\sqrt{T_e-t} + 2nA_{LH}\Big) - \mathcal{N}\Big(A_L - \lambda\sqrt{T_e-t} + 2nA_{LH}\Big)\Big\} \\ &- \sum_{n=-\infty}^{+\infty} e^{\lambda\sqrt{T_e-t}(2A_H+2nA_{LH})} \\ &\times \Big\{ \mathcal{N}\Big(A_K - (2A_H + 2nA_{LH} + \lambda\sqrt{T_e-t})\Big) - \mathcal{N}\Big(A_L - (\lambda\sqrt{T_e-t} + 2A_H + 2nA_{LH})\Big)\Big\}\Bigg] \\ &- S_t \Bigg[\sum_{n=-\infty}^{+\infty} e^{-\lambda\sqrt{T_e-t}(4nA_{LH})+\frac{3}{2}\lambda^2(T_e-t)} \\ &\times \Big\{ \mathcal{N}\Big(A_K - 2\lambda\sqrt{T_e-t} + 2nA_{LH}\Big) - \mathcal{N}\Big(A_L - 2\lambda\sqrt{T_e-t} + 2nA_{LH}\Big)\Big\} \\ &- \sum_{n=-\infty}^{+\infty} e^{\lambda\sqrt{T_e-t}(4A_H+4nA_{LH})+\frac{3}{2}\lambda^2(T_e-t)} \\ &\times \Big\{ \mathcal{N}\Big(A_K - (2A_H + 2nA_{LH} + 2\lambda\sqrt{T_e-t})\Big) - \mathcal{N}\Big(A_L - (2\lambda\sqrt{T_e-t} + 2A_H + 2nA_{LH})\Big)\Big\}\Bigg]\Bigg\}. \end{aligned} \tag{151}$$

8 Fade-in-out options

A *double barrier fade-in option* with fixing date T_F pays off

$$(\phi (S_{T_e} - K))^+ \mathbb{I}_{\{L<\min_{[0,T_e]} S_t \le \max_{[0,T_e]} S_t<H\}} \mathbb{I}_{\{F_L<S_{T_F}<F_H\}}, \tag{152}$$

where K denotes the strike, L and H the lower and higher barrier respectively, and F_L and F_H the lower and higher fixing levels respectively.

8.1 Pricing

To price the option, let us look at its value at the fixing date T_F,

$$v(T_F) = e^{-r_d(T_d-T_F)} v_{DKO}(T_F) \mathbb{I}_{\{F_L<S_{T_F}<F_H\}} \tag{153}$$

with the non-discounted price of the double knockout v_{DKO}. The price for $t < T_F$ is hence

$$\begin{aligned} v(t) &= e^{-r_d(T_d-t)} \mathbb{E}^t \left[(\phi (S_T - K))^+ \mathbb{I}_{\{L<\min_{[0,T_e]} S_t \le \max_{[0,T_e]} S_t<H\}} \mathbb{I}_{\{F_L<S_{T_F}<F_H\}} \right] \\ &= e^{-r_d(T_d-t)} \mathbb{E}^t \Big[(\phi (S_T - K))^+ \mathbb{I}_{\{L<\min_{[t,T_F]} S_{t'} \le \max_{[t,T_F]} S_{t'}<H\}} \times \\ &\quad \mathbb{I}_{\{L<\min_{[T_F,T_e]} S_t \le \max_{[T_F,T_e]} S_t<H\}} \mathbb{I}_{\{F_L<S_{T_F}<F_H\}} \Big] \\ &= e^{-r_d(T_d-t)} \mathbb{E}^t \left[\mathbb{I}_{\{L<\min_{[t,T_F]} S_t \le \max_{[t,T_F]} S_t<H\}} \mathbb{I}_{\{F_L<S_{T_F}<F_H\}} v_{DKO}(T_F) \right] \\ &= e^{-r_d(T_d-t)} \int_{F_L}^{F_H} D(L,H,S_{T_F}) v_{DKO}(T_F) \, dS_{T_F}, \end{aligned} \tag{154}$$

where the density of S_{T_F} not having reached the barriers L, H under Q_{T_F} is given by

$$\begin{aligned} &D(L,H,S_{T_e}) = \\ &e^{-\frac{1}{2}\lambda^2(T_F-t)+\frac{\lambda}{\sigma}\ln\frac{S_{T_F}}{S_t}} \times \\ &\sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{1}{2\sigma^2(T_e-t)} \left(\ln\frac{S_{T_e}}{S_t} + 2n\ln\frac{H}{L} \right)^2 \right) \right. \\ &\left. - \exp\left(-\frac{1}{2\sigma^2(T_e-t)} \left(\ln\frac{H^2}{S_{T_e}S_t} + 2n\ln\frac{H}{L} \right)^2 \right) \right] \mathbb{I}_{\{L<S_{T_e}<H\}} \end{aligned} \tag{155}$$

with

$$\lambda = \frac{\mu}{\sigma} - \frac{\sigma}{2}. \tag{156}$$

9 Slide-in corridor

The slide-in corridor is an option paying

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{L \le S_{T_i} \le H\}} (S_T - K)^+, \tag{157}$$

where K denotes the strike, L and H the lower and higher barrier respectively and $T_i < T_d$ the $N+1$ working dates and delivery date respectively. Let t be the valuation date (horizon) and assume that the premium is paid at the premium value date $t+T_p$. Furthermore, we specify different interest rates for different time intervals, i.e., let

- r_p be the interest rate between t and $t+T_p$,
- r_i be the interest rate between t and T_i,
- r_i^d be the interest rate between T_i and T_d.

Let the spot be modeled by

$$dS_t = \mu_i S_t dt + \sigma_i S_t dW \text{ on } [t, T_i] \tag{158}$$

with $\mu_i \triangleq r_i^d - r_i^f$ and σ_i the forward volatility of the asset on $[t, T_i]$, and

$$dS_t = \mu_i^\dagger S_t dt + \sigma_i^\dagger S_t dW \text{ on } [T_i, T_d] \tag{159}$$

with $\mu_i^\dagger \triangleq r_i^{d\dagger} - r_i^{f\dagger}$ and $\sigma_i^\dagger$ the forward volatility of the underlying on $[T_i, T_d]$.

9.1 Pricing

The theoretical value of the option can be written as

$$\begin{aligned} v(t) &= e^{r_p T_p - r^d(T-t)} I\!E^t \left[\frac{1}{N} \sum_{i=1}^N I\!I_{\{L \le S_{T_i} \le H\}} \left(\phi \left(S_T - K\right)\right)^+ \right] \\ &= e^{r_p T_p - r^d(T-t)} \frac{1}{N} \sum_{i=1}^N I\!E^t \left[I\!E^{T_i} \left[I\!I_{\{L \le S_{T_i} \le H\}} \left(\phi \left(S_T - K\right)\right)^+ \right]\right] \\ &= e^{r_p T_p - r^d(T-t)} \frac{1}{N} \sum_{i=1}^N I\!E^t \left[I\!I_{\{L \le S_{T_i} \le H\}} I\!E^{T_i} \left[\left(\phi \left(S_T - K\right)\right)^+ \right]\right] \end{aligned} \tag{160}$$

with

$$S_{T_i} = S_t e^{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T_i - t) + \sigma_i W_{T_i - t}} \tag{161}$$

and

$$d_L \triangleq \frac{-1}{\sigma_t \sqrt{T_i - t}} \left(\ln\left(\frac{S}{L}\right) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T_i - t)\right), \tag{162}$$

$$d_H \triangleq \frac{-1}{\sigma_i \sqrt{T_i - t}} \left(\ln\left(\frac{S}{H}\right) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T_i - t)\right). \tag{163}$$

The inner expectation can be written in terms of the value function of a vanilla options, i.e.,

$$\begin{aligned} I\!E^{T_i} \left[\left(\phi \left(S_T - K\right)\right)^+ \right] &= \phi S_{T_i} e^{\mu_i^\dagger (T - T_i)} \mathcal{N}\left(\phi d_1(S_{T_i})\right) - \phi K \mathcal{N}\left(\phi d_2(S_{T_i})\right) \\ &= e^{r_i^{d\dagger(T-T_i)}} \mathcal{V}_\phi\left(S_{T_i}, K, (T - T_i)\right) \end{aligned} \tag{164}$$

with

$$d_1(S_{T_i}) \triangleq \frac{1}{\sigma_i^\dagger \sqrt{T - T_i}} \left(\ln\left(\frac{S_{T_i}}{K}\right) + \left(\mu_i^\dagger + \frac{1}{2}{\sigma_i^\dagger}^2\right)(T - T_i)\right), \tag{165}$$

$$d_2(S_{T_i}) \triangleq d_1 - \sigma_i^\dagger \sqrt{T - T_i}, \tag{166}$$

and where $\mathcal{V}(S_{T_i}, K, (T-T_i))$ denotes the value function of a plain vanilla option with spot S_{T_i}, strike K and maturity $(T-T_i)$. To integrate on W_{T_i-t}, let W_{T_i-t} be x and $S_{T_i} = S_{T_i}(x)$. This implies for the value of the slide-in corridor

$$\begin{aligned} v(t) &= e^{r_p T_p - r(T-t)} \frac{1}{N} \sum_{i=1}^{N} \int_{d_L}^{d_H} \Big(\phi S_{T_i}(x) e^{-\mu_i^d (T-T_i)} \mathcal{N}\left(\phi d_1(S_{T_i}(x))\right) - \phi K \mathcal{N}\left(\phi d_2(S_{T_i}(x))\right) \Big) \\ &\times \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx. \end{aligned} \tag{167}$$

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