# FX Greeks

Uwe Wystup

MATHFINANCE

November 6, 2018

## Contents

In this FX column we will cover the different types of Greeks commonly used on the FX Options context. In FX, Greeks can be confusing, because they depend on the quotation of the currency pair as well as the currency in which they are calculated. Furthermore, premium can be included or excluded, smile-effect can be included or not included, and numerical approximations may further add to the confusion. We will now cover all the possibilities step by step.

**Example Market** EUR-USD spot reference $S = S_0$ =1.1000, forward rate $F$ =1.1102, USD money market interest rate $r_d^{MM}$ =1.009%, EUR money market interest rate $r_f^{MM}$ =-0.810%, volatility $\sigma(K)$ =9.784%, ATM volatility=10.493%, 25-delta-Risk-Reversal RR=-2.466%, 25-Delta-Butterfly BF=0.360%. [1]

**Example Product** EUR call USD put $\phi$ =+1, strike $K$ =1.1500, maturity 6 months = 182 days between delivery and spot date, $T_d - T_s = \frac{182}{360}$ =0.5056, $T_e - T_h = \frac{182}{365}$ =0.4986, EUR nominal (call currency amount) $N$ =1,000,000, USD nominal (put currency amount) $NK$ =1,150,000. horizon $T_h$ 5 Dec 2016, spot date $T_s$ 7 Dec 2016, expiry date $T_e$ 5 Jun 2017, delivery date $T_d$ 7 Jun 2017.

# 1 Delta

Delta generally represents the change of the value of a contract based on the change of the underlying spot.

In the case of FX derivatives delta represents the change of the value of a derivative based on the change of the underlying exchange rate. In the following we use the currency pair EUR/USD as an example for the more general notion of FOR/DOM. The exchange rate can be quoted in two directions: EUR-USD denotes the price of one EUR in units of USD; USD-EUR denotes the price of one USD in units of EUR. Additional difficulties arise from the possibility to quote a price of an option in both EUR or USD, and that a delta can refer to both a EUR notional or a USD notional. Moreover, a change in the exchange rate can mean either a change in EUR-USD or a change in USD-EUR. Apart from that, delta can be calculated analytically as a derivative of the pricing formula (if there is one) or numerically by a ratio of finite differences. For the finite differences, there are one-sided and two-sided difference quotients. And for the change of the underlying spot one can assume an absolute change or a relative change. Overall, even in a simple Black-Scholes model, this produces a whole variety of deltas (and other sensitivities). Finally, if we include smile, i.e. a dependence of the volatility on moneyness, the ratio of spot and strike price, we will consider smiled delta and more generally smiled Greeks. Using a pricing model beyond Black-Scholes, such as a stochastic volatility model or jump-diffusion model, will give rise to model delta. And here, we need to make assumptions, whether we re-calibrate a model after a spot shift, or assume model parameters to be constant. Let us denote the value of an FX option by $v(S)$ in USD, with $S$ denoting the current price of one EUR in USD, and by $R = \frac{1}{S}$ the price of one USD in EUR.

## 1.1 Spot-Delta and Forward-Delta

Besides assuming a change of the FX spot price one can also assume a change in the value of a forward contract. This is because, one can delta-hedge the spot change risk of an option in the spot

[1] Data sourced from SuperDerivatives for 5 Dec 2016, except spot reference

market, for which spot delta is required, or in the forward market, for which a forward delta is required. Forward delta differs from the spot delta only by omitting the discount factor in the foreign currency (EUR), more precisely by the EUR discount factor that ensures the interest-rate-forward parity to hold.

We restate the value of an FX option and forward contract in the Black-Scholes model as

$$
\begin{aligned}
v_{opt}(S) &= \phi[Se^{-r_f(T_d-T_s)}\mathcal{N}(\phi d_+) - Ke^{-r_d(T_d-T_s)}\mathcal{N}(\phi d_-)], && (1)\\
v_{fwd}(S) &= \phi[Se^{-r_f(T_d-T_s)} - Ke^{-r_d(T_d-T_s)}], && (2)\\
F &= Se^{(r_d-r_f)(T_d-T_s)} \text{ “forward”}, && (3)\\
d_\pm &= \frac{\ln\frac{F}{K} \pm \frac{1}{2}\sigma^2(T_e-T_h)}{\sigma\sqrt{(T_e-T_h)}}. && (4)
\end{aligned}
$$

As usual we use the put-call-indicator $\phi$, which takes the value of +1 in case of a call option and of -1 in case of a put option. Spot- and forward-delta are defined as

$$
\begin{aligned}
\Delta &= v'(S) = \frac{\partial v_{opt}}{\partial S}, && (5)\\
\Delta_f &= \frac{\partial v_{opt}}{\partial v_{fwd}} = v'(S)e^{r_f(T_d-T_s)}, && (6)
\end{aligned}
$$

and hence differ only by the factor $e^{r_f(T_d-T_s)}$; which is the artificially calculated discount factor in the foreign currency (EUR), whose only purpose it is to satisfy the interest-rate-forward parity (3).

## 1.2 Delta-Premium-Included and Premium-Excluded

Spot-delta $v'(S)$, in our example 31.84%, is the percentage of the EUR notional a trader would buy in the spot market to delta-hedge a sold option. The EUR amount to buy is obviously $Nv'(S)$ =318,400. This assumes the premium of the option is paid in USD. If the premium is paid in EUR, the EUR amount to buy must be reduced by exactly this premium. We obtain

$$
\begin{aligned}
\Delta_{ex} = \Delta &= v'(S), && (7)\\
\Delta_{in} &= v'(S) - \frac{v(S)}{S}. && (8)
\end{aligned}
$$

A risk manager of a EUR-based bank will naturally be concerned with the change of the option's value measured in EUR, as the EUR-USD rate changes. Defining the option value in EUR as $w(S) = \frac{v(S)}{S}$, this change can be calculated via

$$
\begin{aligned}
\frac{\partial}{\partial S}w(S) &= \frac{\partial}{\partial S}\frac{v(S)}{S} = \frac{v'(S)S - v}{S^2}\\
&= \frac{1}{S}\left[v'(S) - \frac{v(S)}{S}\right] = \frac{\Delta_{in}}{S}. && (9)
\end{aligned}
$$

The risk manager's delta is closely related to the premium-adjusted delta of a trader. For a EUR-based trader the delta hedge amount in USD is $\mathrm{d}(v/S)/\mathrm{d}(1/S)$. This turns out to be $s(v' - v/S)$ USD or equivalently $v' - v/S$ EUR. More details on delta- and ATM conventions can be found in [?] and [?].

### 1.3 Discrete Delta for a Relative Spot Change

It is common practice in risk management to calculate delta numerically via a quotient of finite symmetric differences. This is mainly because delta is calculated on the portfolio level rather than for single transactions. For delta we commonly assume a *relative/percentage* change of the FX spot by $\alpha = 1\%$, which means we consider the ratio

$$\begin{aligned} v(S+\alpha/2S) - v(S-\alpha/2S) &= \frac{v(S+\alpha/2S) - v(S-\alpha/2S)}{(S+\alpha/2S) - (S-\alpha/2S)} \cdot \alpha S \\ &= \frac{v(S+\alpha S) - v(S-\alpha S)}{\alpha S} \cdot \alpha S \\ &\approx v'(S) \cdot \alpha S = \Delta_{ex} \cdot \alpha S. \end{aligned} \tag{10}$$

This represents the change of the USD value of the option. Similarly, the change of the EUR value of the option is given by

$$\begin{aligned} w(S+\alpha/2S) - w(S-\alpha/2S) &\approx w'(S) \cdot \alpha S \\ &= \Delta_{in} \cdot \alpha. \end{aligned} \tag{11}$$

Therefore, the risk managers' delta assuming a relative change of the FX spot, and assuming risk is measured in EUR, agrees exactly with the premium-adjusted or premium-included delta of an FX options trader, in the case the option premium is paid in EUR, (up to the factor $\alpha$).

### 1.4 Delta for a Change of the Inverse FX Spot

The value of the option $v$ as a function of the inverse FX spot $1/S$, can be written as $v\left(\frac{1}{1/S}\right)$. Now, shifting the reciprocal spot $1/S$ rather than the spot $S$, we obtain the relation

$$\begin{aligned} v\left(\frac{1}{\frac{1}{S} + \frac{\alpha}{2}\frac{1}{S}}\right) - v\left(\frac{1}{\frac{1}{S} - \frac{\alpha}{2}\frac{1}{S}}\right) &\approx -v'(S) \cdot \alpha S \cdot \frac{1}{1-(\alpha/2)^2} \\ &\approx -v'(S) \cdot \alpha S \end{aligned} \tag{12}$$

for small $\alpha$. This means that it is reasonable to approximate delta calculated as a relative shift of the spot by the negative delta calculated as a relative shift of the inverse spot. However, note that these two versions of delta are not strictly equal.

### 1.5 Relation of Delta to Notional Amounts

Spot-delta $Nv'(S)$ denotes how many EUR to buy when delta-hedging a short option. Thus, $NSv'(S)$ denotes the amount of USD to sell. Since the USD notional is $NK$, we obtain the USD amount to sell per USD notional of the option as $Sv'(S)/K$. Equivalently we need to buy $-Sv'(S)/K$ USD per USD notional. Similar results can be derived for premium-adjusted delta.

## 1.6 Smiled Delta and Model Delta

When including the volatility smile, but still using the Black-Scholes formula for valuation, as it is common market practice, we obtain smile-adjusted Greeks. While ATM Greeks use the ATM volatility for the calculation of Greeks, smiled Greeks used the volatility for the given strike and then interpolation model in place, and model Greeks take the entire model into account, i.e. use the volatility for the given strike and the slope of the volatility curve. On the one hand the value of an option $v$ depends directly on the spot $S$, but on the other hand it also depends on the volatility, which is itself a function of the spot and strike. Therefore, the *model delta* must be calculated as

$$\frac{\mathrm{d}}{\mathrm{d}S}v(S,\sigma(S,K)) = v_S + \mathsf{VanillaVega}\cdot\frac{\partial\sigma(S,K)}{\partial S}, \tag{13}$$

where the first term is the *smiled delta* and the second term is the well-known *windmill-adjustment*. Using homogeneity $\sigma(aS,aK)=\sigma(S,K)$ (see [?]), we know that

$$\frac{\partial\sigma(S,K)}{\partial S} = -\frac{\partial\sigma(S,K)}{\partial K}\frac{K}{S}. \tag{14}$$

The slope of the implied volatility on the strike space leads to smile-adjusted valuation of digital options, and similarly to smile-adjusted delta of standard options. The slope is not a consequence of the Black-Scholes model, in fact, it contradicts the Black-Scholes model. It depends on the method chose to interpolate and extrapolate the smile. Consequently, smiled delta and model delta are not uniquely determined, but depend on the smile construction procedure in place. Note that all the other variants of delta discussed so far can also be ATM deltas or smiled.

## 1.7 Model Delta for Parabolic Interpolation on the Delta-Space

As an (educational!) example, we consider the parabolic interpolation of implied volatility on the forward delta space on the brought up by Malz [?], and its derivative

$$\tilde\sigma(\Delta_f) = ATM - 2RR(\Delta_f - 50\%) + 16BF(\Delta_f - 50\%)^2, \tag{15}$$

$$\tilde\sigma'(\Delta_f) = -2RR + 32BF(\Delta_f - 50\%). \tag{16}$$

We conclude

$$\begin{aligned}\frac{\partial\sigma(S,K)}{\partial K} &= \frac{\partial}{\partial K}\tilde\sigma(\Delta_f(K,\sigma(K))) &(17)\\ &= \tilde\sigma'(\Delta_f)\left[\frac{\partial}{\partial K}\Delta_f(K) + \frac{\partial\Delta_f}{\partial\sigma}\frac{\partial\sigma}{\partial K}\right]\\ &= [-2RR + 32BF(\Delta_f - 50\%)]\left[\mathcal{N}'(d_+)\frac{-1}{K\sigma\sqrt{T_e - T_h}} + \mathcal{N}'(d_+)\frac{-d_-}{\sigma}\frac{\partial\sigma}{\partial K}\right],\end{aligned}$$

where the equation contains the term of interest $\frac{\partial\sigma}{\partial K}$ on both sides, and can be isolated resulting in

$$\frac{\partial \sigma}{\partial K} = \frac{-\tilde{\sigma}'(\Delta_f)\mathcal{N}'(d_+)}{K\sqrt{T_e - T_h}[\sigma + \tilde{\sigma}'(\Delta_f)\mathcal{N}'(d_+)d_-]}, \tag{18}$$

$$\Delta_f = \mathcal{N}(d_+). \tag{19}$$

In our example we obtain a value of -0.13424 and thus for $\frac{\partial \sigma}{\partial S}$ a value of 0.1403.

## 1.8 Delta Variants Example

As a summary we list the various delta variants in Table 1. All deltas are calculated with the volatility at strike $K$, thus all deltas are smiled. The TV-delta using the ATM volatility would be $v'(S) = 33.18\%$. Most variants can also be calculated using this TV-delta as a basis.

| Formula | Value | Description |
|---|---:|---:|
| $v'(S, \sigma = ATM)$ | 33.18% EUR | TV spot delta: EUR to buy |
| $v'(S)$ | 31.84% EUR | spot delta: EUR to buy |
| $v'(S)e^{r_f(T_d - T_s)}$ | 31.71% EUR | forward delta |
| $-v'(S)S$ | -35.02% USD per EUR | spot delta: USD to buy per EUR notional |
| $-v'(S)S/K$ | -30.46% USD | spot delta: USD to buy per USD notional |
| $v'(S) - v/S$ | 30.47% EUR | premium-adjusted spot delta |
| $(v'(S) - v/S)e^{r_f(T_d - T_s)}$ | 30.33% EUR | premium-adjusted forward delta |
| $v'(S)S - v$ | -33.51% USD pro EUR | p.-a. spot delta: USD to buy per EUR notional |
| $(v'(S)S - v)/K$ | -29.14% USD | p.-a. spot delta: USD to buy per USD notional |
| $Nv'(S)\alpha S$ | 3,502 USD | value change (analytic) as EUR-USD changes |
| $N\left(v(S + \frac{\alpha}{2}S) - v(S - \frac{\alpha}{2}S)\right)$ | 3,504 USD | value change for EUR-USD shift |
| discrete | -3,505 USD | value change for USD-EUR shift |
| $N(v'(S) - v/S)\alpha$ | 3,047 EUR | value change (analytic) as EUR-USD changes |
| discrete | 3.047 EUR | value change for EUR-USD shift |
| discrete | -3.049 EUR | value change for USD-EUR shift |
| $v'(S) + v_\sigma \frac{\partial \sigma(S,K)}{\partial S}$ | 35.74% EUR | corresponding model spot delta |

**Table 1:** *Variants of Delta with $\alpha = 1\%$ Relative Shift; Option Value is 151 USD pips or 1.37%EUR. All Deltas are Smiled Except TV Spot Delta; Model Delta is Calculated via Parabolic Interpolation on the Forward-Delta Space $\tilde{\sigma}(\Delta_f) = ATM - 2RR(\Delta_f - 50\%) + 16BF(\Delta_f - 50\%)^2$. Windmill-Adjustment 3.90%.*

I would particularly like to point out the difference of a spot delta, which is calculated with the smile volatility for the given strike, and the model spot delta, which also takes into account the slope of the smile at the strike. If the interpolator is the assumed to be the right choice, then this is the delta hedge to actually perform. In our example, missing out the windmill effect, the delta hedge would be wrong by about 4% of the EUR notional. However, the market quotes from common providers typically use forward delta (with volatility at strike, but without the slope). This is because windmill-adjustment depends on the choice of the interpolator, and there is no market consensus in place for such a choice. The windmill-adjustment (see [?], p. 80) can be calculated analytically for the parabolic smile interpolation, which is useful to illustrate the effect, but by no means the market standard, just to make that clear.

# 2 Gamma

Gamma defined as change of delta as spot changes. This leads to a variety of variants for gamma, as a natural consequence of the delta variants.

## 2.1 Analytic Gamma

Mathematically gamma is given by

$$\frac{\partial^2 v}{\partial S^2} = e^{-r_f(T_d - T_s)} \mathcal{N}'(d_+) \frac{1}{S\sigma\sqrt{T_e - T_h}} = 4,7070. \tag{20}$$

Its unit is $\text{EUR}^2/\text{USD}$. The second derivative can be approximated by a butterfly

$$\frac{\partial^2 v}{\partial S^2} \approx \frac{v(S+h) - 2v(S) + v(S-h)}{h^2}. \tag{21}$$

For $h = S \cdot 1\%$ we obtain 4.7052.

## 2.2 Traders' Gamma

As a trader one would commonly consider the change of delta as spot changes relatively by 1%. We obtain

$$\begin{aligned} \Delta(S + 0,5\%S) - \Delta(S - 0,5\%S) &= \frac{\Delta(S + 0,5\%S) - \Delta(S - 0,5\%S)}{1\%S} \cdot 1\%S \\ &\approx \Delta_S \cdot 1\%S = 51,777 \text{ EUR}, \end{aligned} \tag{22}$$

or 56,955 USD. The symmetric difference turns out to be 51,738 EUR or 56,912 USD. To compare, SuperDerivatives shows 51,779 EUR or 56,957 USD.

## 2.3 Smiled and Model Gamma

Taking into account the smile effect we can calculate as we did for the Model Delta (13)

$$\begin{aligned} \frac{\partial^2}{\partial S^2} v(S, \sigma(S,K)) &= v_S + \text{VanillaVega} \cdot \frac{\partial^2 \sigma(S,K)}{\partial S^2} \\ &+ \text{VanillaVanna} \cdot \frac{\partial \sigma(S,K)}{\partial S}, \end{aligned} \tag{23}$$

which yields

$$\frac{\partial^2 \sigma(S,K)}{\partial S^2} = \frac{\partial^2 \sigma(S,K)}{\partial K^2} \frac{K^2}{S^2} + \frac{\partial \sigma(S,K)}{\partial K} \frac{2K}{S^2} \tag{24}$$

based on Equation (14).

**Model Gamma for Parabolic Interpolation on the Delta Space** As an example we consider again the parabolic interpolation on the forward delta space and obtain

$$\frac{\partial^2 \sigma(S,K)}{\partial K^2} = 3.3835, \tag{25}$$

$$\frac{\partial^2 \sigma(S,K)}{\partial S^2} = 3.4430. \tag{26}$$

# 3 Vega

Vega is defined as the change of the derivative's value as volatility changes.

## 3.1 Analytic Vega

Mathematically, vega is given by

$$\frac{\partial v}{\partial \sigma} = Se^{-r_f(T_d - T_s)}\sqrt{T_e - T_h}\mathcal{N}'(d_+) = 0.2774. \tag{27}$$

This is well-defined in case of a model with one volatility $\sigma$, e.g. in a Black-Scholes model with constant volatility. For models with smile we commonly use *aega*, representing the change of value as the ATM volatility changes, *rega* or *revga*, representing the change of value as the risk reversal changes, and *sega* or *bufga*, representing the change of value as the butterfly changes.

## 3.2 Traders' Vega

A trader will typically consider vega as the change of the USD or EUR value of a derivative contract (or a book of derivatives) assuming a 1% absolute/constant change in volatility.

$$\begin{aligned} v(\sigma + 0,5\%) - v(\sigma - 0,5\%) &= \frac{v(\sigma + 0,5\%) - v(\sigma - 0,5\%)}{1\%} \cdot 1\% \\ &\approx v_\sigma \cdot 1\%. \end{aligned} \tag{28}$$

The change in value is thus $Nv_\sigma \cdot 1\%$ =2,778 USD or 2,526 EUR. A calculation via finite differences would yield 2,778 USD 2,525 EUR. To compare, SuperDerivatives shows $2,775$ USD. I used symmetric difference quotients but depending on the risk management system or its settings one-sided difference quotients are also in use. In particular, we do want to ensure that for very small volatilities as e.g. in (USD-SAR) the volatilities for symmetric differences don't turn negative. The choice of the constant 1% is common, but not mandatory.

# 4 Summary

The list clearly shows that any discussion about Greeks will take a long time, and the longer you think about it, the worse it gets. Actually, you shouldn't have read this.

# References