How can a 50/50 bet have odds of 1:2 instead of 1:1?

Uwe Wystup, MathFinance AG, Frankfurt am Main

A digital call (put) option with a strike at-the-money (ATM) pays 1 unit of currency if the price of the underlying is at or above (below) the strike at maturity. Ignoring a drift (caused by the forward curve), one would expect a profit to be 100%: 50% stake plus 50% if the option is in the money. Digitals trade on many underlyings including FX, commodities, equities, precious metals, stock indices, even a geometric Brownian motion. Nowadays they are even available to retail investors on bank platforms or even non-bank vendors.

Today we attack the following common questions:

- Q1: What is the initial price of the digital option given the potential profit is 50%, corresponding to on-odds of 1:2?
- Q2: Is that price consistent with the Black/Scholes (or small-step-standard-binomial) model?
- Q3: Which simple model can explain a 50% potential profit?

- A1: A pay-off 1unit of domestic currency is the only case with a positive rate of return, so the option price p must solve (1-p)/p = 50%, i.e. p = 2/3=67%, which is significantly higher than 50%, which is what one might expect for a 50/50-type of a bet. A bid-offer spread alone unlikely explains this difference.

- A2: In such models the price of an ATM digital option goes to 0.5 as time to expiry goes to 0. So, the answer is no, unless we admit completely unrealistic and explosive scenarios for interest rates and volatility.

- A3: So how do we explain this? A price (in the sense of MTM) of 67% for an ATM spot digital (paying DOM, domestic currency) is easily possible. When you calculate the Black-Scholes value with the ATM volatility you would get 50%. Note that the ATM strike K_{ATM} in the FX options market is given by $F\times e^{-0.5\sigma^2 T}$ for all currency pairs with premium-included delta convention, where F denotes the forward price, σ the ATM-volatility and T the time to maturity in years. The value of the digital call in the Black-Scholes model is known to be $e^{-rT}\times N(d_-)$ in foreign currency, where r represents the foreign continuous interest rate, N the cumulative normal distribution function evaluated at $d_- = \frac{\ln\frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$, which is zero for K= K_{ATM}, and consequently yields a Black-Scholes value of $e^{-rT}\times 50\%$.

  So far the theoretical value (TV), the value traders use as a checksum and which is generally the value of a derivative contract in the Black-Scholes model with a flat ATM volatility. In practice, derivatives are priced with smile, which leads to the so-called windmill effect for digitals.

  The *windmill adjustment* (Wystup, 2017) is – VanillaVega multiplied by the slope of the smile on the strike space $\sigma'(K)$. The function $\sigma(K)$ is exhibited in Figure 1.

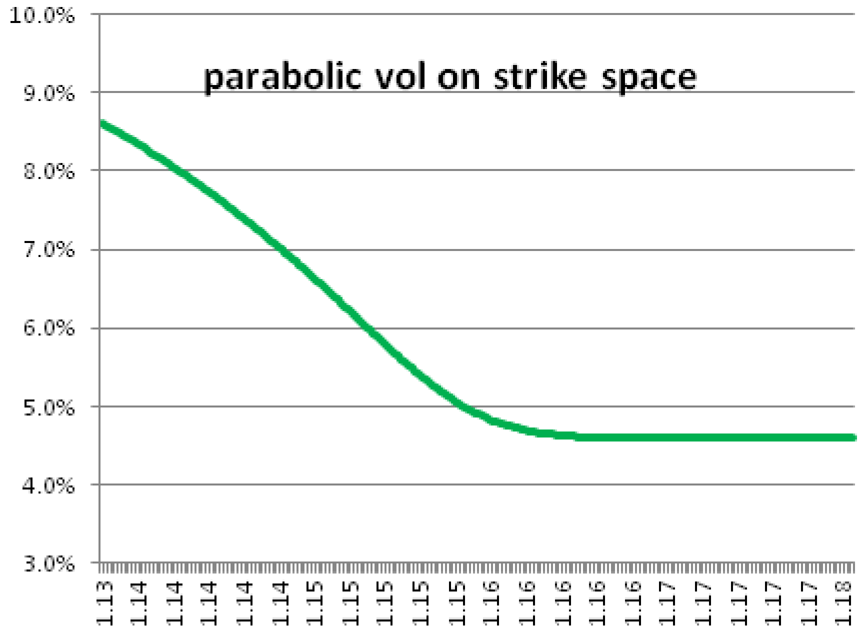

Figure 1: example of a volatility interpolation on the strike space (x-axis) with implied volatilities on the y-axis. Input data is 6% for ATM, 3% for risk reversals in favor of put, 0.4% for butterfly. This is derived from the parabolic volatility interpolation on the forward delta space illustrated in Figure 3.

Since VanillaVega is always positive, the sign of the adjustment is opposite to the slope of the volatility smile (viewed on the strike space); thus for a negative risk reversal, the windmill will push the price up. We consider the example of USDCHF, with the market expectation with the fear of a declining USD, which is typically reflected by a negative risk reversal. For a ten-day digital call wind blows about 16% upwards; hence a 66% mid price is perfectly market consistent. I am using rates close to zero for CHF and USD, so forward = spot, ATM volatility of 6%, risk reversal -3%, butterfly 0.4% for the sake of example. This is a simplified but realistic market scenario. With such a large *relative risk reversal* (risk reversal divided by the ATM volatility), the windmill-adjustment of 16%, yielding a mid price of 66%, an interbank offer price of 67% is easily explained. This corresponds to on-odds of 1:2, or a potential profit of 50% respectively. The simple Black-Scholes model along with the good old parabola know from middle school can explain it all. Figure 2 illustrates how this smile adjustment affects the probability density of the final spot price. We observe that with smile effect taken into account there is more mass for higher spot prices, thus explaining the value of a digital call exceeding 50% significantly.

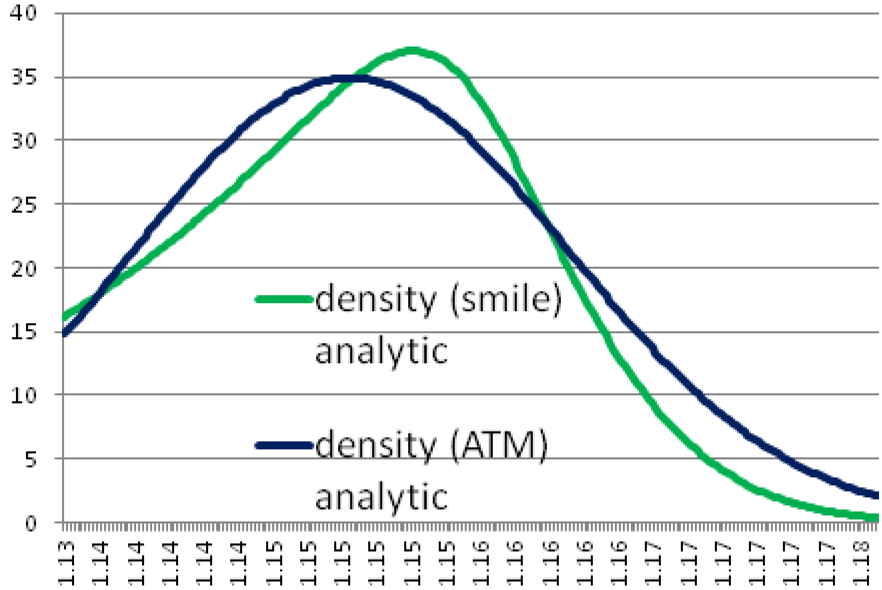

Figure 2: Impact of smile effect on the probability density of the final spot price on the x-axis. The density (ATM) analytic is the log-normal density of the Black-Scholes model. The density with smile effect is derived from the volatility smile exhibited in Figure 1.

How does it work as time to maturity goes to zero? Can we still have deviations of the same size?

Let's consider a popular educational model for smile interpolation: the parabola suggested by A. Malz (Malz, 1997), which is a function of volatility on the forward delta call space, known as $\sigma(\Delta)$=**ATM**-2**RR**(Δ-50%)+16**BF**(Δ-50%)^2, where Δ denotes the forward call delta, **RR** the 25-delta risk reversal, **BF** the 25-delta butterfly, illustrated in Figure 3.

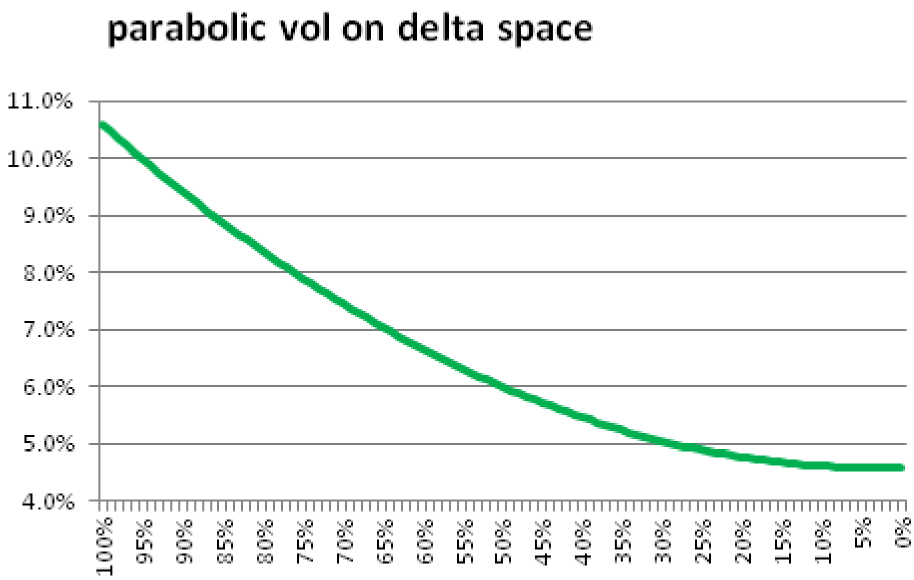

Figure 3: example of a parabolic volatility interpolation on the forward delta call space (x-axis) with implied volatilities on the y-axis. Input data is 6% for ATM, 3% for risk reversals in favor of put, 0.4% for butterfly.

Here is your job for the weekend: assuming **ATM**, **RR**, **BF** stays the same as time to maturity goes to zero, verify that under this parabolic smile model, the limiting windmill adjustment is -**RR**/(π×**ATM**) for the delta-neutral ATM strike.

If **RR** is -1.5% and **ATM** 10%, then this windmill adjustment is roughly 5%, which would bring the price up from a theoretical value of 50% to a market value of 55%. A higher risk reversal of -3% for an ATM 6% causes an even higher up-wind of +15.92%, which is very close to the upwind of 16% in the ten-day contract.

This might be much harder to explain with more sophisticated models based on jump-diffusion or stochastic volatility.

Note, that for very short maturities, this is a purely theoretical result, because for O/N options vega is essentially not tradable, and the smile surface will very likely look different from the 1M or 1W tenor.

For example, EUR-USD data of 30 March 2017 for the tenors, O/N (red), 1W (pink), 1M (olive) with a cubic spline interpolation indicates how different market data can look like for the very short term. In Figure 4 we show implied volatility on the delta-space (and obviously do not want to argue that cubic splines are necessarily the best choice of interpolating).

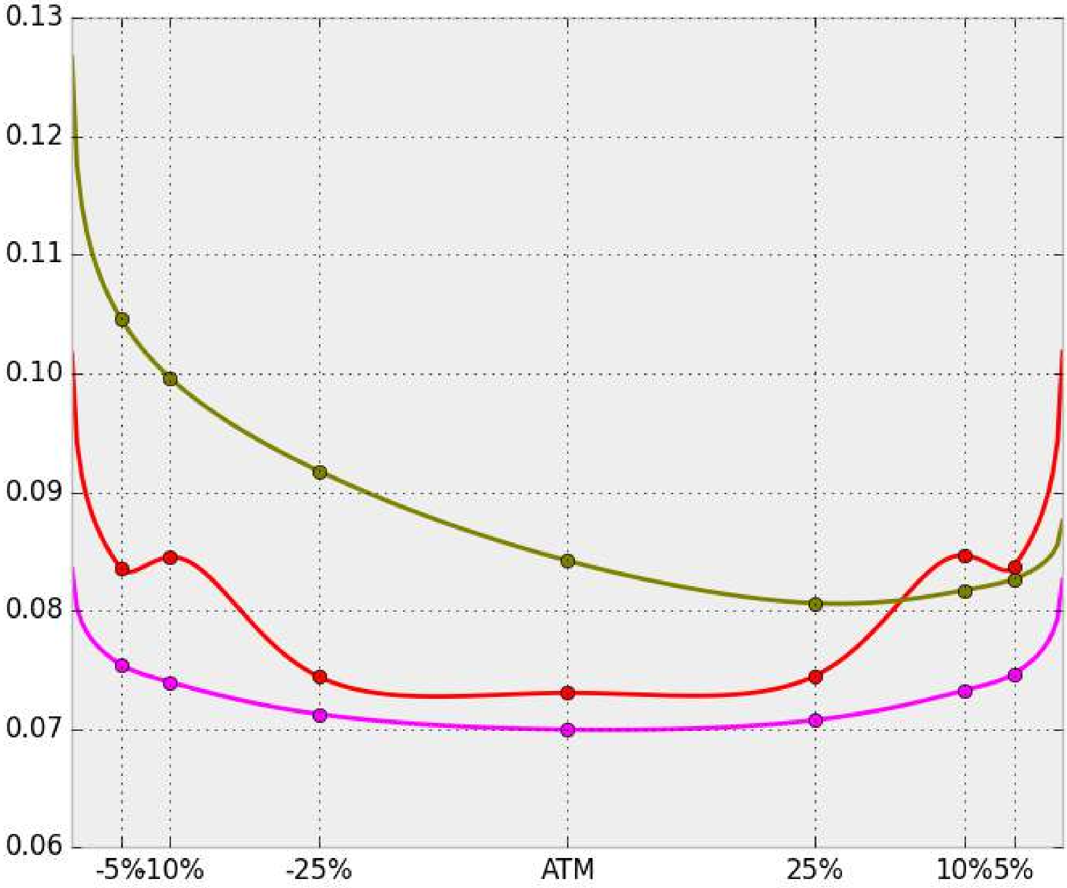

Figure 4: Input volatilities (sourced from Digital Vega) shown as dots for three tenors O/N, 1W, 1M, with possible interpolation methods. The x-axis has put forward deltas on the left, call forward deltas on the right. The 5-delta volatilities for the (red) O/N tenor appear too low and can cause a challenge for volatility interpolation.

The volatility surface is the key object for option pricing, not only vanilla options, but also exotic options, such a simple digital. We noticed that the slope of the smile on the strike space is one of the essential drivers for smile-adjustments, even for simple exotics. This slope in turn depends on the interpolation/extrapolation method. Therefore, putting a curve through a few points can turn into a challenging project as the results in the valuation of your exotics book can be substantial, leave alone risk parameters.

Bibliography

Malz, A. M. (1997). Option-Implied Probability Distributions and Currency Excess Returns. *FRB of New York Staff Report No. 32*. doi: Malz, Allan M., Option-Implied Probability Distributions and Currency Excess Returns (November 1997). FRB of New York Staff Report No. 32. Available at SSRN: https://ssrn.com/abstract=943500 or http://dx.doi.org/10.2139/ssrn.943500

Wystup, U. (2017). *FX Options and Structured Products, 2nd Edition.* Chichester: Wiley.