Arbitrage in the Perfect Volatility Surface

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Constructing the FX volatility surface is an ongoing challenge in the derivatives industry, even for quants – or should I say, only for quants, because I am quite sure the vast majority of the population couldn't care less? Starting from at-the-money volatilities (ATM), risk reversals (RR) for 25- and 10-delta, butterflies (BF) or strangles for 25- and 10-delta, we need to find a smile curve, and taking term structure into account, a volatility surface that prices these quoted market instruments correctly and satisfy a number of conditions including but not limited to:

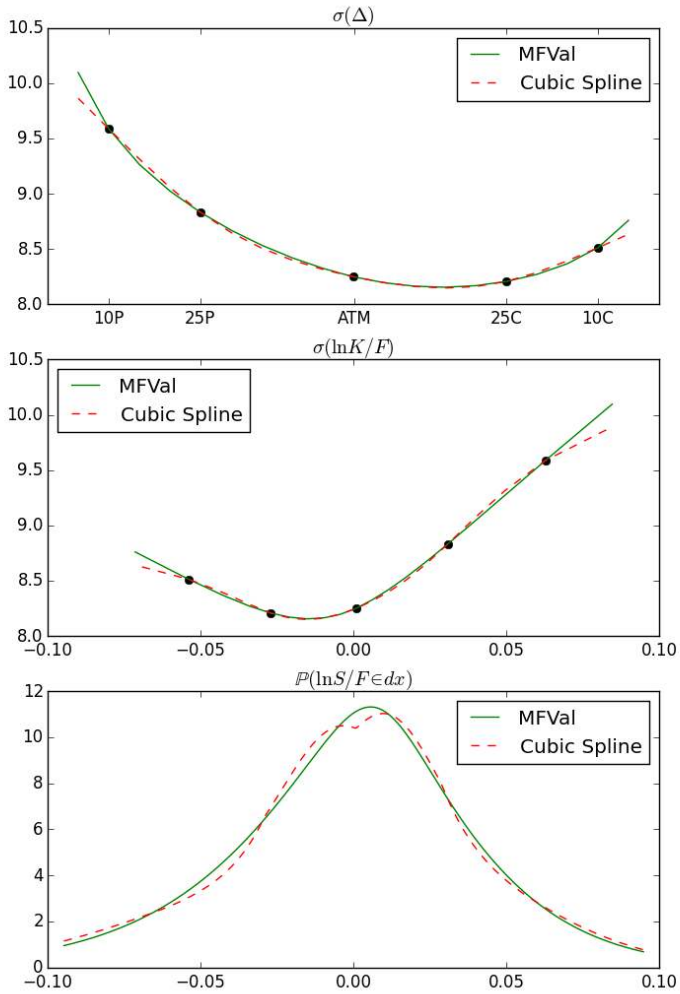
1. Smoothness, ideally up to second order: to construct a local volatility model;
2. Reasonable slope on the wings: to match prices of variance swaps;
3. Reasonably rich to reflect negative market strangle quotes;
4. Extendable to a consistent spreading logic;
5. Free of arbitrage

The smile construction interacts with the choice of interpolation and extrapolation. The following methods have reasonably sized fan-clubs:

1. Cubic splines: easy to do, common technique, fits all the points;
2. Polynomials including the parabola on the forward delta space proposed by Malz 1997;
3. Vanna-volga for interpolation proposed by Castagna/Mercurio;
4. Kernels;
5. Stochastic volatility models, most prominently SABR and its numerous patchwork extensions for wings;
6. Stochastic volatility inspired (SVI) parametric fits proposed by Gatheral.

While it will take a long time to check and vet all the methods, today, I would like to direct your attention to a few highlights, aiming to correct some common misconceptions.

1) The first figure visualizes the differences between the popular cubic spline interpolation and a version of SVI labeled as "MFVal" in the graph. We observe that there is actually some concavity in market volatilities when going far OTM and plotting implied volatility on the log-moneyness space instead implied volatility on the delta space. The non-smooth shape of the implied density is a consequence of cubic spline being smooth up-to second order only (i.e. 3rd derivative may be discontinuous).



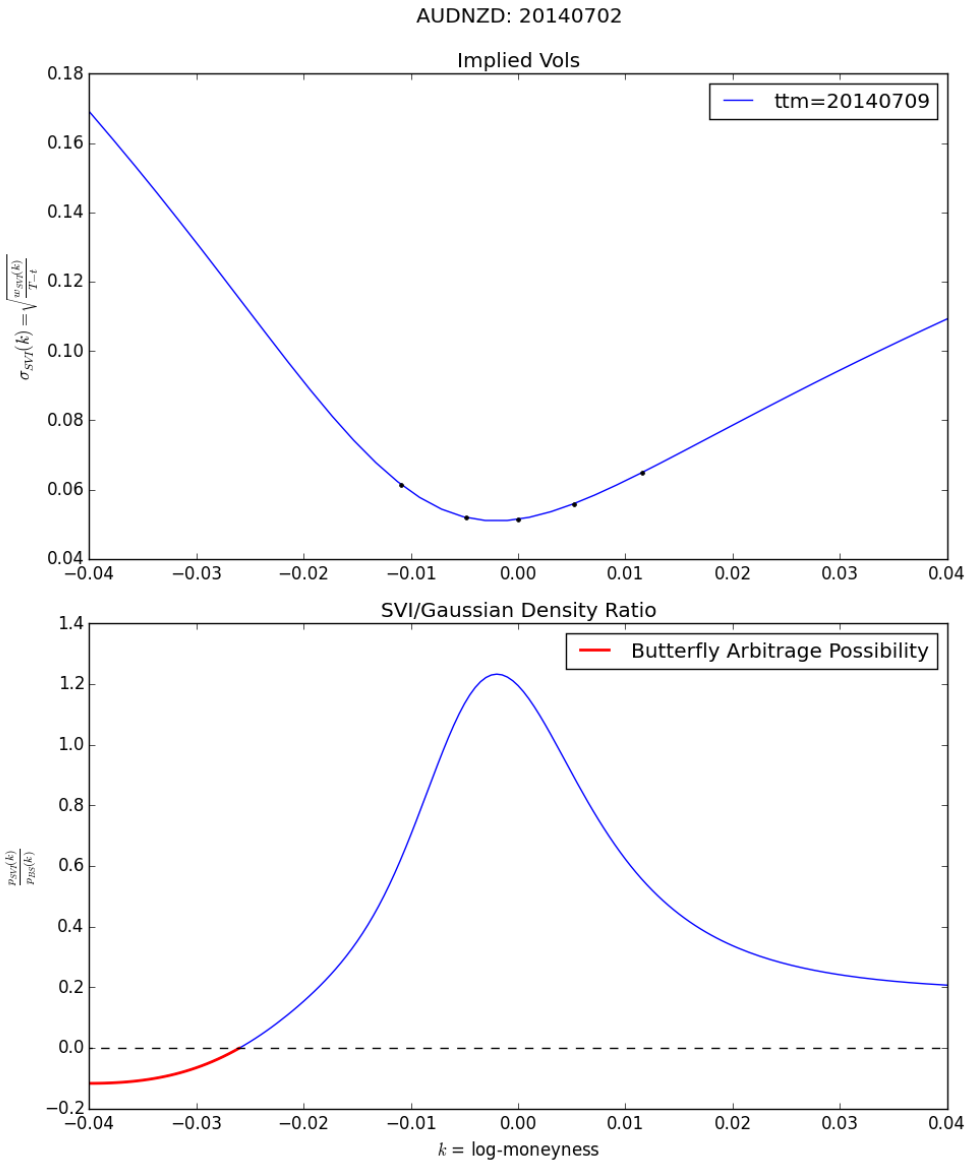
We show EUR/USD volatility interpolation of 31 October 2016 for the 3M tenor. Top: implied volatilities on the delta space; middle: implied volatilities on the log-moneyness space; bottom: implied probability density of the final spot price on the log-moneyness space.

In a sense the fit using cubic splines is too good: matching all the points can lead to non-unimodal densities – look at the dent in the density near the 0.00 point –, even if the fit of the smile curve passes the *mere-eye-visual-inspection-sanity-check*.

While this is probably not dangerous and not an obvious arbitrage violation, it still doesn't feel right. In particular, it might causes P&L shifts for digital options and other exotics.

2) Since cubic splines might overdo the fit, another fan club in the market uses SVI and their various offspring, a parametric curve inspired from stochastic volatility models, an idea going back to an idea by Jim Gatheral. Here, we can match many desirable conditions imposed on the smile curve, and it will at the very least usually look nice.
However, it is not necessarily arbitrage-free. In fact: even a nice fit that passes the *mere-eye-visual-inspection-sanity-check*, with supposedly steep slopes on the wings may not be free of

arbitrage. Consider the example of a **Butterfly Arbitrage** possibility on SVI smile for AUD/NZD 1W options on 2014-Jul-02[1]. We see perfect fit, but as shown in the graph, the low delta put side admits Butterfly Arbitrage or equivalently, implies a negative probability density of the 1W spot, indicated in red in the lower graph lower LHS.



Conclusion: Visual inspection of the smile fit helps, and should be done if possible, but even that plus a perfect fit of the 5-point market do not guarantee unimodal densities or arbitrage-free smile surfaces. If one has to automate the pricing or even the valuation of FX options using contributed

[1] Spot reference = 1.0784, fwd = 1.07845, AUD discount factor = 0.999712587139, 10-delta RR = 0.35%, 25-delta RR = 0.40%, ATM vol = 5.14%, 25-delta BF = 0.25%, 10-delta BF = 1.175%

market data, one needs to generate an overlay of techniques to avoid surface artifacts, at least to a range of options indicated as tradable.