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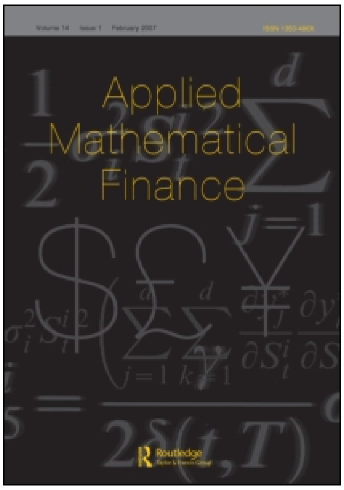

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Closed Formula for Options with Discrete Dividends and Its Derivatives

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Closed Formula for Options with Discrete Dividends and Its Derivatives

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ABSTRACT *We present a closed pricing formula for European options under the Black–Scholes model as well as formulas for its partial derivatives. The formulas are developed making use of Taylor series expansions and a proposition that relates expectations of partial derivatives with partial derivatives themselves. The closed formulas are attained assuming the dividends are paid in any state of the world. The results are readily extensible to time-dependent volatility models. For completeness, we reproduce the numerical results in Vellekoop and Nieuwenhuis, covering calls and puts, together with results on their partial derivatives. The closed formulas presented here allow a fast calculation of prices or implied volatilities when compared with other valuation procedures that rely on numerical methods.*

KEY WORDS: Equity option, discrete dividend, hedging, analytic formula, closed formula

1. Introduction

1.1 Motivation

The motivation to return to this issue is the fact that whenever a new product, model or valuation procedure is developed, the problem that arises with discrete dividends is dismissed or overlooked by applying the usual approximation that transforms the discrete dividend into a continuous stream of dividend payments proportional to the stock price. After all that has been said about the way to handle discrete dividends, there are still strong reasons to justify such an approach.

We here recall the reasons that underlie the use of this method by the majority of market participants and pricing tools currently available. We choose the word *method* instead of *model* although one could look for what model would justify such calculations and find the Escrowed Model, as it is known in the literature. We do not follow this reasoning because we consider that such a model would be unacceptable since it admits arbitrage. The reason being that such a model would imply two different diffusion price processes for the same underlying stock under the same measure if two options were considered with different maturities and spanning over a different number of dividend

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payments. We thus refuse the model interpretation and consider the procedure that replaces the discrete dividend into a continuous stream of dividend payments as an approximation to the price of an option, under a model that remains arbitrage free when several options coexist. The models in Section 1.2 belong to that class.

The drivers behind the huge popularity of this method are mostly due to the (i) tractability of the valuation formulas, (ii) applicability to any given model for the underlying stock, and (iii) the preserved continuity of the option price when crossing each dividend date.

However, the method has some significant drawbacks. First and foremost, no proof has ever been present that this method would yield the correct result under an acceptable model in the sense above. In fact, for the natural extensions to the Black–Scholes (BS) model described in Section 1.2, the error grows larger as the dividend date is farther away from the valuation date. This is exactly the opposite behaviour of what one would expect from an approximation – a larger period of time between the valuation date and the dividend date implies that the option valuation functions are smoother and thus easier to approximate. The other side of the inaccurate pricing coin is the fact that this method does not provide a hedging strategy that will guarantee the replication of the option payoff at maturity. To sum up, no numerical procedure based on this method returns (or converges to) the true value of the option, in any of the acceptable models we are aware of. It still seems like the advantages outweigh the drawbacks since it is the most widely used method.

An example may help to demonstrate this. Consider a stochastic volatility model with jumps. Now consider the valuation problem of an American style option under this model. The complexity of this task is such that a rigorous treatment of discrete dividends, i.e. a modification of the underlying's diffusion to account for that fact, would render the model intractable.

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1.2 Description of the Problem

In the presence of discrete dividend payments, diffusion models like the BS model are no longer an acceptable description of the stock price dynamics. The risks that occur in this context are mainly the potential losses arising from incorrect valuation and ineffective hedging strategy. We address both of these issues in this article.

The most natural extension to a diffusion model to allow for the existence of discrete dividends is to consider the same diffusion, for example

$$dS_t = S_t(r\,dt + \sigma\,dW_t), \tag{1}$$

and add a negative jump with the same size as the dividend, on the dividend-payment date as $S_{t_D} = S_{t_D^-} - D$. This gives rise to the new model diffusion

$$dS_t = S_t(r\,dt + \sigma\,dW_t) - D\mathbb{I}_{\{t \geq t_D\}}, \tag{2}$$

where S is the stock price, r is the constant interest rate, σ is the volatility and W is a standard Brownian motion. $S_{t_D^-}$ refers to the time immediately before the dividend-payment moment, t_D, and S_{t_D} to the moment immediately after.

There are though some common objections to this formulation. A first caveat may be the assumption that the stock price will fall by the amount of the dividend size. This objection is mainly driven by the effects taxes have on the behaviour of financial agents and thus market prices. We will not consider this objection in this article and thus assume Model (2) to be valid. A second objection may be that the dividend-payment date and amount are not precisely known until a few months before their payment. We also believe this to be the case, but a more realistic model in this respect would significantly grow in complexity. Our goal is rather to devise a simple variation that can be applied to a wide class of models that does not worsen the tractability of the model and produces accurate results.

Finally, the model admits negative prices for the stock price S. This is in fact true and can easily be seen if one takes the stock price $S_{t_D^-}$ to be smaller than D at time t_D^-. A simple solution to this problem is to add an extra condition in Equation (2) where the dividend is paid only if $S_{t_D^-} > D$, i.e.

$$\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma\mathrm{d}W_t) - D\,\mathbb{I}_{\{t \geq t_D\}}\mathbb{I}_{\{S_{t_D^-} > D\}}. \tag{3}$$

However, in most practical applications, this is of no great importance as the vast majority of the companies that pay dividends have dividend amounts that equal a small fraction of the stock price, i.e. less than 10% of it, rendering the probability assigned to negative prices very small. For this reason we may drop this condition whenever it would add significant complexity.

In the remainder of this section we review the existing literature on the subject and the reasons that underlie the use of the method most popular among practitioners. We then turn to develop the formulas in Section 2, and in Section 3 we reproduce the numerical results in Vellekoop and Nieuwenhuis (2006) together with put prices and partial derivatives. Section 4 concludes.

1.3 Literature Review

Here we shortly review the literature on modifications of stock price models coping with discrete dividend payments. Merton (1973) analysed the effect of discrete dividends in American calls and states that the only reason for early exercise is the existence of unprotected dividends. Roll (1977), Geske (1979) and Barone-Adesi and Whaley (1986) worked on the problem of finding analytic approximations for American options. John Hull (1989) in the first edition of his book establishes what was to be the most used method to cope with discrete dividends. The method works by subtracting from the current asset price the net present value of all dividends occurring during the life of the option. On the other end of the spectrum, Musiela and Rutkowski (1997) propose a model that adds the future value at maturity of all dividends paid during the lifetime of the option to the strike price. To balance these two last methods, Bos and Vandermark (2002) devise a method that divides the dividends in 'near' and 'far' and subtracts the 'near' dividends from the stock price and adds the 'far' dividends to the strike price. A method that considers a continuous geometric Brownian motion with jumps at the dividend-payment dates is analysed in detail by Wilmott (1998) by means of numerical methods. Berger and Klein (1998) propose a

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non-recombining binomial tree method to evaluate options under the jump model. Bos *et al.* (2003) devise a method that adjusts the volatility parameter to correct the subtraction method stated above. Haug *et al.* (2003) review existing methods' performance and pay special attention to the problem of negative prices that arise within the context of the jump model and propose a numerical quadrature scheme. Björk (1998) has one of the clearest descriptions of the discrete dividends problem for European options and provides a formula for proportional dividends. Shreve (2004) also states the result for proportional dividends. Vellekoop and Nieuwenhuis (2006) described a modification to the binomial tree method to account for discrete dividends preserving the crucial recombining property.

2. Closed Formula

The derivation of the closed formula assumes a BS model as in Equation (1) with constant interest rate r and constant volatility σ. However, the following can be easily modified to allow for time-dependent volatility. Furthermore, our arguments consider and are only valid for European-style options.

We assume a problem with n dividends D_i, with $i = 1, \ldots, n$, having their respective payment dates on t_i ordered in this manner $t_0 < t_1 < \ldots < t_n < T$ and having t_0 and T as the valuation date and the option's maturity date, respectively.

We take a vanilla call option as our working example. We start by focusing on the time point just after the last dividend payment, which we will refer to as t_n. We choose this point in time because it is the earliest moment on which we can make a conjecture with respect to the price of the option, i.e. from this point on, we know how to price and hedge a claim, for there are no dividends left until the option matures. The price of our call would thus be a function $C(S_{t_n}, t_n)$[1] of the stock price and time, the celebrated BS formula that we state here for completeness sake:

$$C(S_t, t) = S_t N(d_+) + K\mathrm{e}^{-r(T-t)} N(d_-),$$

$$d_\pm = \frac{\log(\frac{S_t}{K}) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{4}$$

As usual, K and T are the strike price and the maturity date, respectively. As it is going to be used extensively throughout this article, we take here the opportunity to also present the general formula for the ith derivative with respect to the first variable, S_t, of Formula (4) developed by Carr (2001)

$$\partial_1^i C(S_t, t) = S_t^{-i} \sum_{j=1}^{i} \mathcal{S}_1(i,j)\delta^j,$$

$$\delta^j = S_t N(d_+) + K\mathrm{e}^{-r(T-t)} \frac{N'(d_-)}{\sigma\sqrt{T-t}} \sum_{h=0}^{j-2} \frac{H_h(d_-)}{\left(-\sigma\sqrt{T-t}\right)^h}, \tag{5}$$

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where $N'(x)$ denotes the probability density function of the standard normal distribution, $\mathcal{S}_1(i,j)$ the Stirling number of the first kind and $H_i(d)$ are Hermite polynomials.

The problem we face now is how to move one step back in time to $t < t_n$. For that we take Assumption (2) in Section 1.2. This assumption yields $C(S_{t_n}, t_n) = C(S_{t_n^-} - D_n, t_n)$, and its right-hand side already refers to the stock price at a time point just before t_n. We now wish to move further back in time but still without crossing any other dividend date, that is, to t_{n-1}. This task is a straightforward application of option pricing theory yielding

$$e^{-r(t_n - t_{n-1})}\mathbb{E}^Q_{t_{n-1}}\left[C\left(S_{t_n^-} - D_n, t_n\right)\right], \tag{6}$$

which is the discounted expectation of the random variable $C(S_{t_n^-} - D_n, t_n)$ under the risk-neutral measure Q with respect to the σ-algebra $\mathcal{F}_{t_{n-1}}$.

Unfortunately, Expression (6) is not directly solvable into a closed formula for it includes the random variable $\log\{S_{t_n^-} - D_n\}$, which has no known or explicit distribution.

At this point our hope is to replace the formula $C(S_{t_n^-} - D_n, t_n)$ by an equivalent representation that would not involve $\log\{S_{t_n^-} - D_n\}$. The natural candidate is the Taylor series expansion of C taken at the point $S_{t_n^-}$ and with a shift of size $-D_n$. Unfortunately, we know from the works of Estrella (1995) that such replacement is not valid for all values of $S_{t_n^-}$, and thus

$$C\left(S_{t_n^-} - D_n, t_n\right) \neq \sum_{i=0}^{\infty} \frac{(-D_n)^i}{i!}\partial_1^i C\left(S_{t_n^-}, t_n\right). \tag{7}$$

The reason for this is the fact that the Taylor series expansion of the BS formula for calls is convergent only for shifts of a size smaller than $S_{t_n^-}$ and diverges otherwise. In our case, where $S_{t_n^-} < D_n$ the Taylor series does not produce the same values as $C(S_{t_n^-} - D_n, t_n)$, and in turn, the expectation in (6) will also be affected. We acknowledge, though, that the risk-neutral probability of $S_{t_n^-} < D_n$ is very small, and thus the effect in (6) of the divergence of the Taylor series for the values $S_{t_n^-} < D_n$ will not affect our approximation too much.

Confronted with this result, we tried to carry the derivation forward on rigorous grounds, rewriting Expression (6) by introducing an indicator function for the set $A = \{\omega : S_{t_n^-} > D_n\}$,

$$e^{-r(t_n - t_{n-1})}\mathbb{E}^Q_{t_{n-1}}\left[C\left(S_{t_n^-}, t_n\right) + \left(C\left(S_{t_n^-} - D_n, t_n\right) - C\left(S_{t_n^-}, t_n\right)\right) \cdot \mathbb{I}_A\right]. \tag{8}$$

This approach did lead to a closed formula for the case of problems with only one dividend payment (see Veiga and Wystup, 2007). However, the cost of this rigour was a highly complex formula that cannot be generalized to fit multiple dividend problems. For this reason we here take a different route.

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We will consider that

$$C\left(S_{t_n^-} - D_n, t_n\right) \approx \sum_{i=0}^{\eta_n} \frac{(-D_n)^i}{i!} \partial_1^i C\left(S_{t_n^-}, t_n\right) \tag{9}$$

with η_n high enough to approximate $C\left(S_{t_n^-} - D_n, t_n\right)$ reasonably well, for all $S_{t_n^-}$. We thus trade the error of this approximation for the tractability that it enables. We do so because we believe that in almost all realistic scenarios the error is not significant. In fact, our results in Section 3 based on this assumption do provide very good results with scenarios even more demanding than realistic market conditions.

Hence, we rewrite Expression (6) as

$$\mathrm{e}^{-r(t_n - t_{n-1})} \mathbb{E}_{t_{n-1}}^{Q} \left[\sum_{i=0}^{\eta_n} \frac{(-D_n)^i}{i!} \partial_1^i C\left(S_{t_n^-}, t_n\right) \right]. \tag{10}$$

Since we have a finite series as integrand function, we can safely interchange the integral with the summation, yielding

$$\mathrm{e}^{-r(t_n - t_{n-1})} \sum_{i=0}^{\eta_n} \frac{(-D_n)^i}{i!} \mathbb{E}_{t_{n-1}}^{Q} \left[\partial_1^i C\left(S_{t_n^-}, t_n\right) \right]. \tag{11}$$

Finally, to turn Expression (11) above into an explicit formula we use the following:

Proposition 2.1. Let $C(S_t, t)$ and all its derivatives be continuous functions in its first variable, then

$$\mathbb{E}_t^{Q}\left[\partial_1^i C(S_{t_k}, t_k)\right] = \mathrm{e}^{-\left(r + \frac{1}{2} i \sigma^2\right)(i-1)(t_k - t)} \partial_1^i C\left(S_t \mathrm{e}^{-i\sigma^2 (t_k - t)}, t\right). \tag{12}$$

Proof. We prove the proposition by mathematical induction. For $i = 0$ we get

$$C(S_t, t) = \mathrm{e}^{-r(t_k - t)} \mathbb{E}_t^{Q}[C(S_{t_k}, t_k)], \tag{13}$$

which is true, for it states that the discounted BS price is a martingale under the risk-neutral measure Q.

Now we need to check that the proposition for i implies the same proposition for $i + 1$. Changing variables by $P_t = S_t \mathrm{e}^{-i\sigma^2 (t_k - t)}$ we get

$$\mathbb{E}_t^{Q}\left[\partial_1^i C(S_{t_k}, t_k)\right] = \mathrm{e}^{-\left(r + \frac{1}{2} i \sigma^2\right)(i-1)(t_k - t)} \partial_1^i C(P_t, t), \tag{14}$$

where now $S_{t_k} = P_t \exp\left\{ \left(r - \frac{\sigma^2}{2} + i\sigma^2\right)(t_k - t) + \sigma(W_{t_k} - W_t) \right\}$.

We differentiate the left-hand side with respect to P_t, and since $\partial_1^n C$ is continuous for all $n \in \mathbb{N}$, we apply Leibniz integral rule, getting

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$$\int_{\mathbb{R}} \partial_1^{i+1} C\left(P_t \mathrm{e}^{\left(r-\frac{\sigma^2}{2}+i\sigma^2\right)(t_k-t)+z\sigma\sqrt{t_k-t}}, t_k\right) \mathrm{e}^{\left(r-\frac{\sigma^2}{2}+i\sigma^2\right)(t_k-t)+z\sigma\sqrt{t_k-t}-\frac{1}{2}z^2} \mathrm{d}z/\sqrt{2\pi}. \tag{15}$$

Taking $z = y + \sigma\sqrt{t_k - t}$ leaves us with

$$\mathrm{e}^{(r+i\sigma^2)(t_k-t)} \int_{\mathbb{R}} \partial_1^{i+1} C\left(P_t \mathrm{e}^{\left(r-\frac{\sigma^2}{2}+(i+1)\sigma^2\right)(t_k-t)+y\sigma\sqrt{t_k-t}}, t_k\right) \mathrm{e}^{-\frac{1}{2}y^2} \mathrm{d}y/\sqrt{2\pi}, \tag{16}$$

which equals

$$\mathrm{e}^{(r+i\sigma^2)(t_k-t)} \mathbb{E}_t^Q \left[\partial_1^{i+1} C\left(\mathrm{e}^{\sigma^2(t_k-t)} S_{t_k}, t_k\right)\right] \tag{17}$$

with $S_{t_k} = P_t \exp\left\{\left(r - \frac{\sigma^2}{2} + i\sigma^2\right)(t_k - t) + \sigma(\widetilde{W}_{t_k} - \widetilde{W}_t)\right\}$ and $\widetilde{W}$, a Brownian motion.

Taking the derivative of the right-hand side of (14) with respect to P_t, we obtain

$$\mathrm{e}^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_k-t)} \partial_1^{i+1} C(P_t, t). \tag{18}$$

Equating the derivatives of both sides of (14) with respect to P_t, i.e. (17) and (18), rearranging and using the fact $P_t = \mathrm{e}^{\sigma^2(t_k-t)} S_t \mathrm{e}^{-(i+1)\sigma^2(t_k-t)}$, we get

$$\mathbb{E}_t^Q \left[\partial_1^{i+1} C\left(\mathrm{e}^{\sigma^2(t_k-t)} S_{t_k}, t_k\right)\right] = \mathrm{e}^{-\left(r+\frac{1}{2}(i+1)\sigma^2\right)i(t_k-t)} \partial_1^{i+1} C\left(\mathrm{e}^{\sigma^2(t_k-t)} S_t \mathrm{e}^{-(i+1)\sigma^2(t_k-t)}, t\right), \tag{19}$$

which is exactly the claim for $i + 1$ with a positive factor multiplying the first argument of C on both sides of the equation.

We can now write Equation (19) considering a new initial stock price $S_t' = \mathrm{e}^{\sigma^2(t_k-t)} S_t$ and rely on the geometric nature of the diffusion S_t to have also $S_{t_k}' = \mathrm{e}^{\sigma^2(t_k-t)} S_{t_k}$, then yielding

$$\mathbb{E}_t^Q \left[\partial_1^{i+1} C\left(S_{t_k}', t_k\right)\right] = \mathrm{e}^{-\left(r+\frac{1}{2}(i+1)\sigma^2\right)i(t_k-t)} \partial_1^{i+1} C\left(S_t' \mathrm{e}^{-(i+1)\sigma^2(t_k-t)}, t\right). \tag{20}$$

□

Hence, to get a closed formula for a call option maturing at T with one discrete dividend payment at time t_n of amount D_n, we explicitly rewrite Expression (11) and we denominate as $C_n(S_{t_{n-1}}, t_{n-1})$,

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$$\sum_{i=0}^{\eta_n} \frac{(-D_n)^i}{i!} \mathrm{e}^{-\left(r+\frac{1}{2}(i-1)\sigma^2\right)i(t_n-t_{n-1})} \partial_1^i C\left(S_{t_{n-1}} \mathrm{e}^{-i\sigma^2(t_n-t_{n-1})}, t_{n-1}\right). \tag{21}$$

In what follows, we will require a more condensed notation, so we introduce the abbreviations below and suppress the time variable from all C functions.

$$f_{t_n}^h = \exp\left\{-(r+\frac{1}{2}(h-1)\sigma^2)h(t_n - t_{n-1})\right\} \tag{22}$$

$$g_{t_n}^j = \exp\{-j\sigma^2(t_n - t_{n-1})\} \tag{23}$$

Now, Formula (21) becomes

$$C_n(S_{t_{n-1}}) = \sum_{i=0}^{\eta_n} \frac{(-D_n)^i}{i!} f_{t_n}^i \partial_1^i C\left(g_{t_n}^i S_{t_{n-1}}\right). \tag{24}$$

We can now resume our movement backwards in the time axis using the same programme that led us here, namely, apply (2) to move over the dividend date t_{n-1}; apply Approximation (9) now for C_n yielding[2]

$$\begin{aligned} C_n\left(S_{t_{n-1}^-} - D_{n-1}\right) &\approx \sum_{j=0}^{\eta_{n-1}} \frac{(-D_{n-1})^j}{j!} \partial_1^j C_n\left(S_{t_{n-1}^-}\right) \\ &= \sum_{j=0}^{\eta_{n-1}} \frac{(-D_{n-1})^j}{j!} \frac{\partial^j}{\partial^j S_{t_{n-1}^-}} \left\{\sum_{i=0}^{\eta_n} \frac{(-D_n)^i}{i!} f_{t_n}^i \partial_1^i C\left(g_{t_n}^i S_{t_{n-1}^-}\right)\right\} \\ &= \sum_{j=0}^{\eta_{n-1}} \sum_{i=0}^{\eta_n} \frac{(-D_{n-1})^j}{j!} \frac{(-D_n)^i}{i!} f_{t_n}^i g_{t_n}^{ij} \partial_1^{i+j} C\left(g_{t_n}^i S_{t_{n-1}^-}\right); \end{aligned}$$

take the discounted expectation under the measure Q with respect to the σ-algebra $\mathcal{F}_{t_{n-2}}$ and apply Proposition 2.1 to solve the expectation and get

$$C_{n-1}(S_{t_{n-2}}) = \sum_{j=0}^{\eta_{n-1}} \sum_{i=0}^{\eta_n} \frac{(-D_{n-1})^j}{j!} \frac{(-D_n)^i}{i!} f_{t_{n-1}}^{i+j} f_{t_n}^i g_{t_n}^{ij} \partial_1^{i+j} C\left(g_{t_{n-1}}^{i+j} g_{t_n}^i S_{t_{n-2}}\right). \tag{25}$$

Running this programme for all n dividends returns the formula for an arbitrary number of dividend payments

$$C_1(S_{t_0}) = \sum_{i_1=0}^{\eta_1} \cdots \sum_{i_n=0}^{\eta_n} \prod_{j=1}^{n} \frac{(-D_j)^{i_j}}{i_j!} f_{t_j}^{I_j} \prod_{k=j+1}^{n} \left(g_{t_k}^{I_k}\right)^{i_j} \partial_1^{I_1} C(G_I S_{t_0}), \tag{26}$$

with $I_l = \sum_{m=l}^n i_m$ and $G_I = \prod_{h=1}^n g_{t_h}^{I_h}$.

Before we conclude this section, we remark that even though we developed our analysis focused on a European call, it remains valid for other types of options. In fact,

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the above analysis is valid for all options that satisfy all conditions it involved, namely, European-style options that are priced by only taking expectations under the risk-neutral measure, Approximation (9); $\partial_1^n C$ is continuous for all $n \in \mathbb{N}$ to apply Leibniz integral rule. The existence of a closed formula for an arbitrary derivative of the option price, e.g. Formula (5), greatly accelerates the calculation process. However, the analysis remains valid if the derivatives are replaced by numerical approximations. This alternative may be useful for problems solved by finite difference methods that return a vector of option prices for different stock prices, thus enabling the calculation of numerical derivatives for all the necessary stock price levels.

Therefore, a European put is an example of another option type covered in this analysis and for which a closed formula for an arbitrary derivative is also available in Carr (2001). In Section 3 we also consider European puts and observe that their prices are coherent with the respective call prices.

2.1 The Greeks

A closed formula for the derivative[3] of the option price of arbitrary order is a straightforward application of the chain rule. Thus, for the dth derivative of the call price with one discrete dividend payment we have

$$\partial_1^d C_1(S_{t_0}) = \sum_{i_1=0}^{\eta_1} \cdots \sum_{i_n=0}^{\eta_n} \prod_{j=1}^{n} \frac{(-D_j)^{i_j}}{i_j!} f_{t_j}^{I_j} \prod_{k=j+1}^{n} \left(g_{t_k}^{I_k}\right)^{i_j} (G_I)^d \partial_1^{d+I_1} C(G_I S_{t_0}). \qquad (27)$$

The derivatives with respect to other variables, namely σ and r, require similar derivations that we skip here since they constitute simple calculus exercises. There is one exception worth mentioning though: the theta, i.e. the derivative with respect to valuation time t. The theta can be calculated by making use of the BS partial differential equation, yielding

$$\partial_2^1 C_1(S_t, t) = rC_1(S_t, t) - rS_t \partial_1^1 C_1(S_t, t) - \frac{1}{2} S_t^2 \sigma^2 \partial_1^2 C_1(S_t, t). \qquad (28)$$

3. Results

For ease of reference we reproduce the results stated in Vellekoop and Nieuwenhuis (2006) for European call options with seven discrete dividend payments. The model parameters are set at $S_0 = 100$, $\sigma = 25\%$ and $r = 6\%$. Furthermore, the stock will pay one dividend per year, with each dividend 1 year after the previous, of amount 6, 6.5, 7, 7.5, 8, 8 and 8 for the first 7 years, respectively. We consider three different scenarios of dividend stream payments referenced by the payment date of the first dividend t_1 set at 0.1, 0.5 and 0.9. With respect to the call and put option specifications, we consider three different options, all with 7 years maturity, with strikes of 70, 100 and 130. The calculations reported in Table 1 were performed taking a second-order approximation for each of the dividend payments, i.e. $\eta_1, \ldots, \eta_7 = 2$. This approximation order proved to be very effective in this case, producing price differences of 0.01 in the worst cases when compared to the results reported in Vellekoop and Nieuwenhuis (2006).

Table 1. European calls and puts, $\sigma = 25\%$, $r = 6\%$, $S_0 = 100$, $T = 7$.

Option			Price	Delta	Gamma	Vega	Theta	Rho
$t_1 = 0.1$								
	K = 70	Call	24.8862	70.6821	69.2653	68.9332	–4.9123	216.9129
		Put	13.0212	–29.3179			0.3758	–234.1280
	K = 100	Call	17.4394	56.0090	77.3505	80.7711	–4.7314	191.5356
		Put	25.2859	–43.9910			1.7394	–397.4851
	K = 130	Call	12.4114	43.8271	75.9637	81.9970	–4.2588	160.8653
		Put	39.9693	–56.1729			3.3947	–566.1352
$t_1 = 0.5$								
	K = 70	Call	26.0752	71.1645	66.2195	70.8947	–4.7747	225.5784
		Put	13.2109	–28.8355			0.4534	–238.8582
	K = 100	Call	18.4890	56.9270	74.3512	83.3331	–4.6298	200.6573
		Put	25.3362	–43.0730			1.7811	–401.7592
	K = 130	Call	13.2968	44.9643	73.6551	85.2207	–4.2018	169.9771
		Put	39.8554	–55.0357			3.3917	–570.4191
$t_1 = 0.9$								
	K = 70	Call	27.2117	71.6629	63.4400	72.6905	–4.6496	233.7131
		Put	13.3718	–28.3371			0.5200	–243.4113
	K = 100	Call	19.4905	57.8120	71.6694	85.6678	–4.5390	209.1948
		Put	25.3620	–42.1880			1.8133	–405.9094
	K = 130	Call	14.1419	46.0412	71.6077	88.1568	–4.1517	178.5016
		Put	39.7248	–53.9588			3.3833	–574.5825

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The table displays the price, the first and second derivatives with respect to S, the derivatives with respect to σ, t and r, i.e. delta, gamma, vega, theta and rho. Each row of the table is a set of quantities that relate to one single option. These were calculated together in one single programme run that took between 2.3 and 2.6 hundreds of a second.[4] We note here that a naïve and straightforward implementation of Formula (26) would be very inefficient because of a large number of repetitive calculations contained in it as well as in the BS general derivative Formula (5).

The key to the performance of our implementation is the caching of all quantities that are needed more than once. We start by breaking up the implementation problem in two routines: one to calculate the BS prices and respective derivatives (Formulas (4) and (5)) and the other routine to calculate the price of the option with dividends and its derivatives (Formulas (26) and (27)).

The first routine is implemented as an object that is initialized with all the parameters K, t_0, T, σ, r, $S_t = 1$ and $\eta_{\max}$, which is the maximum derivative order that will be required during the entire calculation. The stock price is set to 1 because it is the only argument that will differ from call to call. The initialization calculates several quantities that will be used repeatedly and stores them in memory together with look up tables for factorials and the numbers $\mathcal{S}_1$, in particular

$$\frac{1}{\sigma\sqrt{T-t}}, \quad \frac{K e^{-r(T-t)}}{\sigma\sqrt{T-t}}, \quad \frac{1}{(-\sigma\sqrt{T-t})^{i\prime}} d_{\pm}.$$

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This object exposes a method that returns all derivatives of the BS formula, from arbitrary order a to b, for a given stock price S_t. This method starts by correcting $d_\pm$ by adding $\log S_t/(\sigma\sqrt{T-t})$, computing $N'(d_-)$ and $N(d_+)$. It moves on to calculate all δ^j's in (5). It starts by filling a vector indexed by j with the summation term[5] followed by the calculation of the δ^j vector. With all δ^j's in place, each derivative is just a sum of the numbers $\mathcal{S}_1$ multiplied by δ^j divided by a power of S_t.

The second routine also stores several quantities that are used repeatedly. It also takes into account that multiplications are more time consuming than summations. Thus, instead of storing all $f_{t_n}^h$ and $g_{t_n}^j$, the routine stores their logarithms. Accordingly, all products of these terms in (26) and (27) are implemented as sums of their logarithms and their final sum is taken through the exponential function. This routine also pre-calculates and stores all $(-D_j)^{i_j}/i_j!$ that involve powers, divisions and factorials that are particularly time consuming. The routine then iterates over all possible combinations of $i_1, \ldots, i_n$. At each combination it calculates the aggregate factor multiplying C and its argument $G_I S_{t_0}$. At this point it is important to note that the argument of C is the same for the price Formula (26) and for derivatives Formula (27), allowing the call of the first routine to request all derivatives between order I_1 and $d + I_1$.[6] The factor multiplying C in the derivatives formula is also very similar to the factor multiplying C in the price formula, differing only by $(G_I)^d$. It is thus quite efficient to calculate the price and all derivatives of interest in the same programme run as their calculations largely overlap.

In any of the cases under scrutiny, to calculate the price, the number of evaluations of the BS pricing formula or any of its derivatives amounts to 2187 or 3^7. In general, the number of evaluations amounts to $\prod_{i=1}^{n}(\eta_i + 1)$.

From the analysis of the table we see that call and put prices are coherent with put–call parity. In every pair we get the relationship $call - put = S - e^{-r(T-t)}K - \sum_{i=1}^{n} e^{-r(T-t_i)} D_i$ and complementary deltas.

Perhaps a more interesting analysis is the comparison of these results with the results produced by *modified stock price* and *modified strike price*. Table 2 shows the same scenarios as above for $t_1 = 0.1$ and only calls using these two methods referred to as *MS* and *MK,* respectively.

It should be noted that all values these methods produce differ rather strongly from the closed formula approximation and from each other. The theta appears to be very problematic as the closed formula value does not stand between both methods' values as is the case for the other quantities. This fact seriously undermines the validity of

Table 2. European calls, $\sigma = 25\%$, $r = 6\%$, $S_0 = 100$, $T = 7$.

Option		Price	Delta	Gamma	Vega	Theta	Rho
$t_1 = 0.1$							
K = 70	MS	20.1576	75.1016	82.8568	48.5396	–4.1634	260.0109
	MK	30.7358	69.9048	52.6414	92.1224	–3.9952	200.4516
K = 100	MS	12.3709	55.5057	103.2509	60.4870	–3.6682	209.8567
	MK	23.1768	58.5707	58.9171	103.1049	–3.9648	193.3094
K = 130	MS	7.7555	39.8123	100.8274	59.0672	–2.9782	158.3466
	MK	17.5976	48.5136	60.2725	105.4769	–3.7385	176.2017

methods that rely on the averaging of these two alternatives. The behaviour of the gamma and vega are also worth noting. These two quantities, that are crucial for the effectiveness of the hedging strategy, show an almost erratic behaviour with one method returning almost twice as much as the other for gamma and vice versa for vega. These are in fact the quantities where the relative differences between the methods and the closed formula are greater.

Finally, one should expect that cases different from the ones here presented may require different approximation orders to achieve convergence of the closed formula. On the one hand, problems with larger individual dividends, with dividends very close to maturity or in the presence of very low volatilities, should require a higher approximation order. The issue at stake is how smooth the function being approximated is – the smoother the function, the lower the required approximation order. On the other hand, the higher the volatility, the greater the probability of having negative stock prices and thus divergence on the Taylor series approximation. To inspect these problems we choose one of the options above, namely, the call with $K = 100$ and $t_1 = 0.1$ and observe how the formula performs on different levels of volatility.

Figure 1 shows, for volatilities from 1% up to 10%, the value calculated by a binomial tree as in Vellekoop and Nieuwennuis (2006), the values for the methods *MS* and *MK* above and the closed formula with all η_is equal to 1, 2, 3 and 4. We can see that the formula with all η_is equal to 2 is effective for volatilities 8% and above, while greater η_is provide good results from 4% on. Below 4% the function is not smooth enough to be approximated, and the results of the formula are far from the binomial tree value.

Figure 2 shows, for volatilities from 30 up to 70%, the same functions. As anticipated, high volatilities will eventually lead the formula to diverge. What we see from the graph, though, is that, for η_is higher than 2, no significant extra precision is obtained but the stability of the formula at lower volatilies is compromised. In this example, with η_is equal to 2, the approximation starts to diverge only after 60%

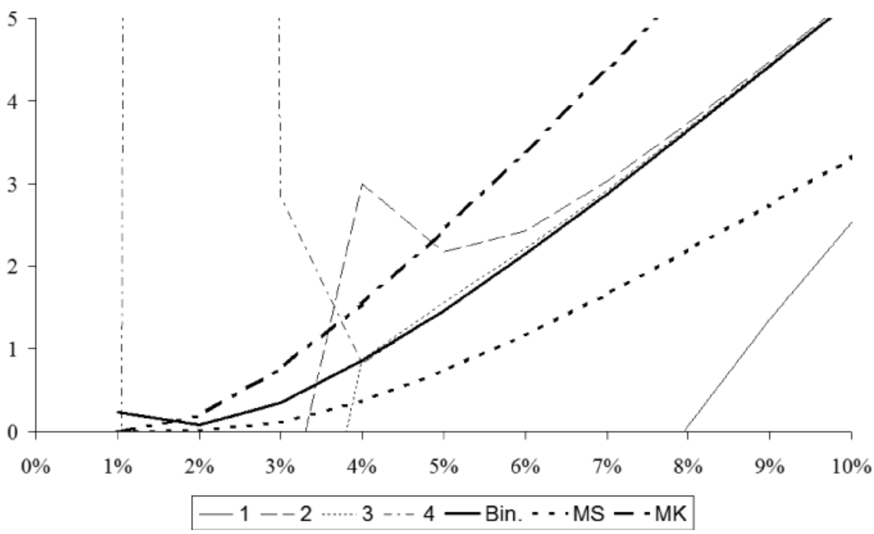

Figure 1. Behaviour in a low volatility environment.

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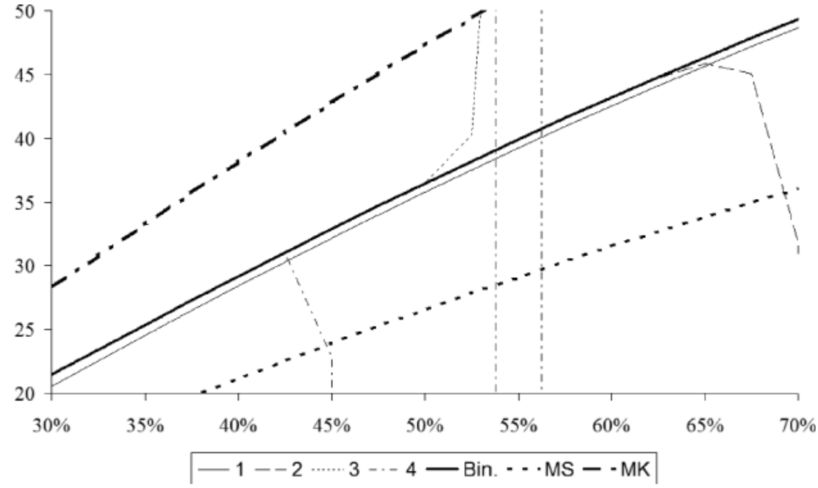

Figure 2. Behaviour in a high volatility environment.

volatility. If η_is equal to 3 or 4 are taken, the approximation diverges at roughly 50 and 40%, respectively. It is also interesting to note that the case with all η_is equal to 1 runs consistently close to the binomial tree value while never being exact. This quantity may be used for control purposes as may the upper and lower bounds given by the methods *MK* and *MS,* respectively.

4. Conclusion

Departing from the well-known behaviour of the option price at the dividend-payment date, we approximate it by Taylor series expansion and successfully manipulate it to arrive at a closed form pricing formula. The formula relies on an approximation and a proposition that relates expectations of partial derivatives with partial derivatives themselves. Besides the pricing formula, we also present formulas for its derivatives with respect to the stock price and with respect to other model parameters.

We present applications of the formulas and successfully reproduce the results for calls reported by Vellekoop and Nieuwenhuis (2006) and calculate the corresponding put options together with the usual hedging quantities. We compare them with the most commonly used pricing methods that rely on modification of the stock price and of the strike price. We observe severe differences both on the pricing and on the hedging quantities.

Our results show that for a setup that is more demanding than usual market conditions, a second-order approximation is fast and sufficient to attain precise results. We also inspect the performance of the formula in extreme scenarios of very high and low volatilities. On the one hand, low volatilities require higher approximation orders to attain precise results. On the other hand, a second-order approximation seems to be the most appropriate for very high volatilities since it shows signs of breaking down and diverging at higher volatilities compared to higher order approximations without any significant loss of precision.

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Future research should focus on developing rules to control the effectiveness of the formula in extreme scenarios of very high or very low volatility environments. The extension of these results to other models than the BS one and the extension of this approach to multi-asset options are also research topics worth pursuing.

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Notes

[1]We omit the model parameters and the option-specific quantities, like maturity or strike, from the function C to preserve clarity.
[2]Note that $S_{t_{n-1}^-}$ is known at time t_{n-1} and thus $\partial^j/\partial^j S_{t_{n-1}^-}\{\cdots\}$ below is only a derivative with respect to an argument of the function and not a derivative with respect to a stochastic variable.
[3]The derivatives of the option price are usually called *Greeks* because Greek alphabet letters are commonly used to denote them.
[4]Using a C++ '.xll' added to MS Excel03 running on an Intel Core2 4400@2 GHz.
[5]Please note that each element of the vector is just the previous plus an extra term composed by the Hermite polynomial (that should be computed by the well known recursive relation) multiplied by a quantity already available in memory.
[6]And even derivatives of I_1 with respect to r and to σ if the first routine is ready to return them.

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