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FX Column: Calendar arbitrage in the FX volatility surface

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Today I would like to share an observation with you that we discovered when analyzing FX volatility market data.

When we construct the FX volatility smile[1] based on the usual FX brokers' data ATM, 25-delta risk reversal (RR) and butterfly (BF), 10-delta risk reversal and butterfly, there is a decent fan club – even among market makers – that uses an SVI (stochastic volatility inspired)[2] parametric function for the variance v on the log-moneyness space k=ln(K/F), where K denotes the strike and F the FX forward rate respectively:

$$v(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2 + s^2}\right)$$

If you have been to high school, you will recognize this as a hyperbola in the variable k. The five SVI parameters allow quite a bit of flexibility, a for the level, inspired by *ATM* volatility, m for *moving* the hyperbola around the k-axis, b for *butterfly* (convexity), ρ for cor*r*elation (skew, *risk reversal*), s^2 for *sharpness*; remember that a hyperbola is a conic section, which degenerates to a cone for s^2=0.

We consider market data of for AUD/JPY displayed in Table 1.

Tenor	10 RR	25 RR	ATM	25 BF	10 BF
6M	-5.475	-2.725	11.71	0.6	2.525
1Y	-7.1	-3.525	12.44	0.65	2.975
2Y	-11.625	-5.9	13.30	0.6	3.225

Table 1: AUD/JPY brokers' market data of 7 Jan 2014, in %, source: Tullett Prebon

When we generate the volatility smile curves using an SVI parameterization we see the variances vT= $\sigma^2 T$ on the log-moneyness space in ***Figure 1***. The SVI fit is based on minimizing the distance of the market quotes from the hyperbola.

[1] Dimitri Reiswich and Uwe Wystup: FX Volatility Smile Construction, *Wilmott,* Volume 2012, Issue 60, pages 58-69.

[2] James Gatheral: a parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives; Global Derivatives & Risk Management Conference 2004, Madrid.

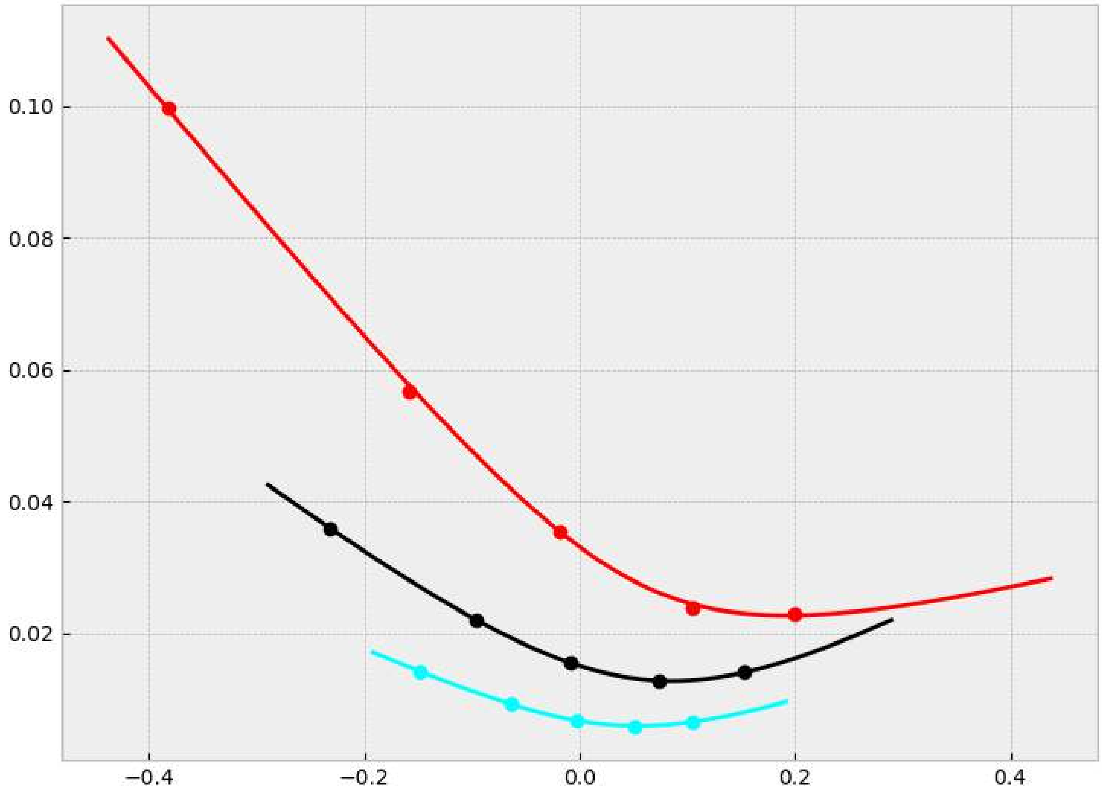

Figure 1: SVI fit to the market data dots from Table 1, log-moneyness k on the x-axis, total variance $\sigma^2 T$ on the y-axis; lowest curve for 6M, middle for 1Y, upper for 2Y.

Calendar arbitrage occurs if the total variance, which is volatility squared times the time to maturity $\sigma^2 T$, for a given moneyness K/F and a longer maturity is smaller than the total variance for the same moneyness (and possibly a strike K adjusted by the ratio of forwards F of the two maturities respectively) and a shorter maturity. Such a scenario would yield negative forward variances. One can visualize calendar arbitrage by checking if the total variance curves intersect. If they do, then there is a (theoretical) calendar arbitrage opportunity. A real arbitrage opportunity would require including bid-offer spreads and a liquid market of very low delta-options.

In the present scenario, we observe a rather good fit of the SVI-curves to the market volatility quotes. They are non-intersecting in the region of quoted volatilities. However, if we zoom out and plot the total variance curves on a wider log-moneyness range (x-axis) in ***Figure 2***, the curves for 2Y (red) and 1Y (black) intersect.

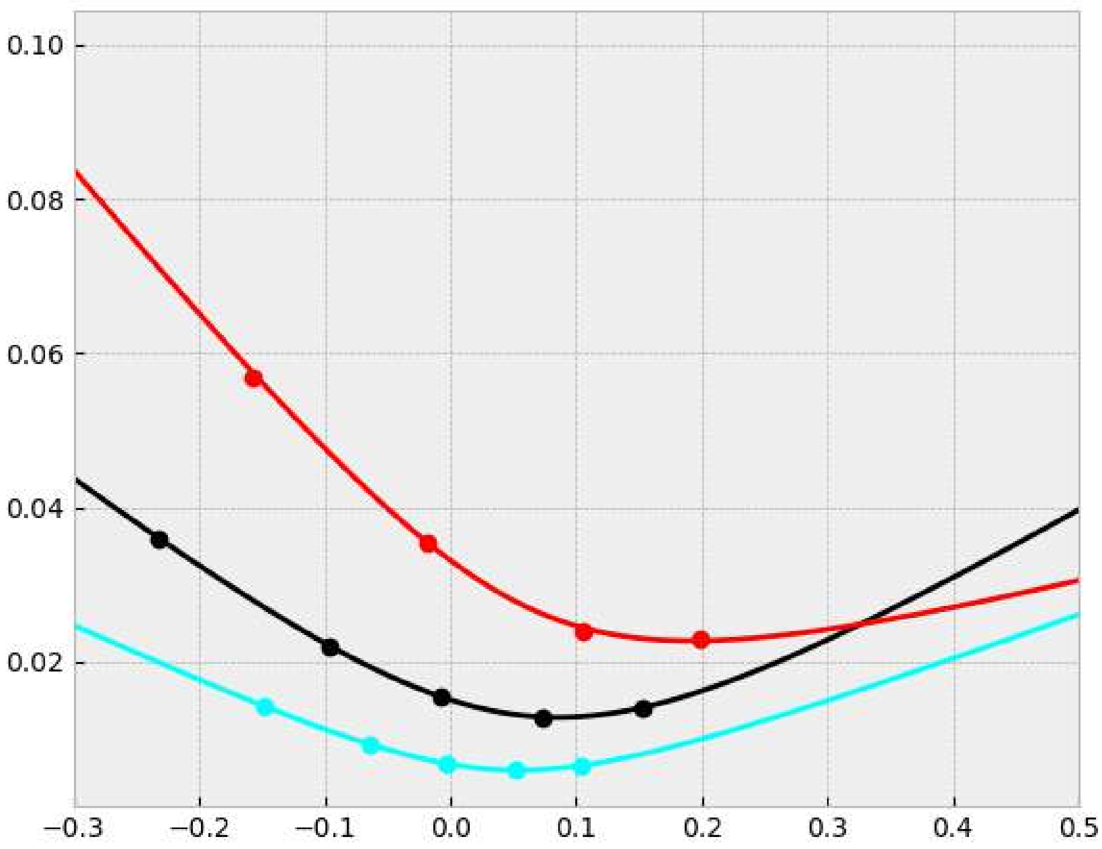

Figure 2: SVI fit to the market data dots from Table 1 zoomed-in, log-moneyness k on the x-axis, total variance $\sigma^2 T$ on the y-axis; lowest curve for 6M, middle for 1Y, upper for 2Y.

While the phenomenon is outside the usual range of traded options, it can still cause problems when using the volatility surface as a building block of an exotics model, such as a local volatility model, where the call options dual theta (derivative with respect to maturity time) is required to be positive after forward correction. Note that this isn't a market-based arbitrage, but a model-based arbitrage. It is the SVI-parameterization that yields this effect. The problem is that each SVI-fit is done for one maturity, and these tenor-wise calibrations don't know of each other.

Solution: We need to impose additional conditions to *control the wings* and hence ensure that calendar arbitrage is prevented in the SVI parameterization, see ***Figure 3***, at least inside a relevant trading range, e.g. between 1% and 99% delta.

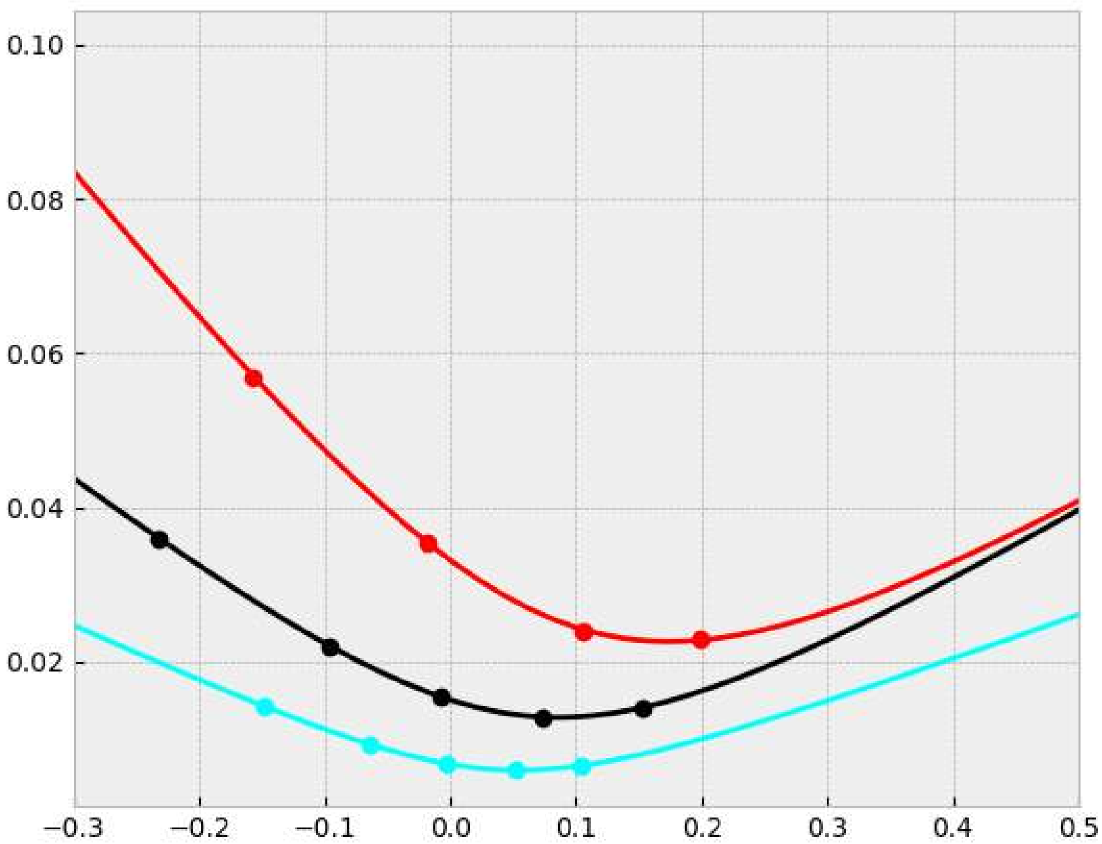

Figure 3: SVI fit to the market data dots from Table 1 with wings controlled, log-moneyness k on the x-axis, total variance $\sigma^2 T$ on the y-axis; lowest curve for 6M, middle for 1Y, upper for 2Y.

The 2Y curve (red) is now on top of the 1Y curve (black). Using this as input for an exotics model is much more suitable.

Summary

Applying various interpolation techniques to construct the FX volatility surface from market quotations may introduce arbitrage, merely by the choice of the interpolation and extrapolation technique. We showed a calendar arbitrage example with the popular SVI-parameterization and an idea how to fix it. Generally volatility surface construction requires detailed attention and permanent inspection.[3]

3 Watch the video on Arbitrage in the Perfect Volatility Surface on https://www.mathfinance.com/wp-content/uploads/2017/03/Arbitrage-in-the-Perfect-Volatility-Surface.mp4