

FX Column: Replication a EUR-paying Double-No-Touch with a Single Double-Knock-Out Option

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In the previous FX Column, we tackled the question of how many exotic options or contracts there are. Today I would like to share a stunning example about the Double-No-Touch contract paying a fixed amount of currency if the spot price S_t observed during the lifetime stays inside a pre-defined range between a lower and higher barrier, L and H .

There are two versions of this touch contract, one that pays a fixed amount of domestic currency (USD in the case of the currency pair EUR-USD), and another one that pays a fixed amount of foreign currency (EUR). In equity markets we commonly distinguish these versions by cash-or-nothing (USD paying) and asset-or-nothing (EUR paying) as for all digital types of payoffs.

Perfect Replication: USD-DNT with Two DKOs

$$\text{USD-paying DNT}(L,H) = \text{DKOCall}(K=L,L,H) + \text{DKOPut}(K=H,L,H)$$

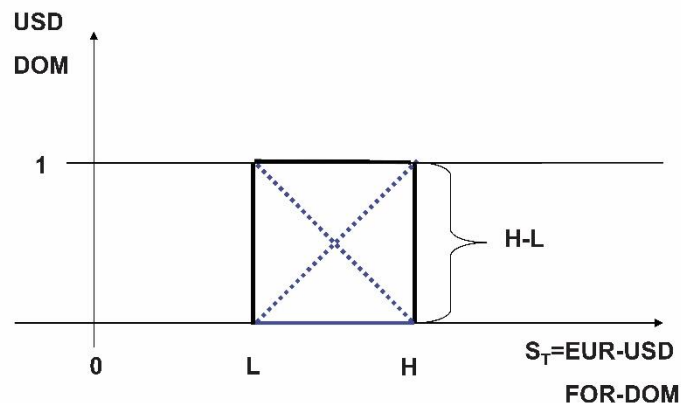


Figure 1: USD-paying Double-No-Touch (DNT) Static Replication with DKO call and DKO Put Options

Revising the USD-paying Double-No-Touch:

As the payoffs (in USD) indicate in Figure 1, one can replicate a long double no-touch with barriers L and H using a portfolio of

1. a long double-knock-out call with barriers L and H and strike L ,
2. a long double-knock-out put with barriers L and H and strike H .

Upon exercise of both options, the payoff would be $(S_T - L) + (H - S_T) = H - L$ USD. Since the payoff is 1 USD, the notional of both options should be $1 / (H - L)$. In which currency? As you may know this my favorite question when teaching FX options ☺.

$$DNT(L, H) = \frac{DKOCall(L, L, H) + DKOPut(H, L, H)}{H - L}$$

Static Replication of a FOR-Paying Double-No-Touch (DNT)

The idea can obviously be recycled by just flipping the quotation of the currency pair, i.e., instead of EUR-USD, we consider the inverted spot USD-EUR, the price of a USD in EUR.

We can replicate the EUR-paying DNT by taking a USD-EUR double-no-touch with lower barrier $1/H$ and higher barrier $1/L$. This double-no-touch can be composed as before using

$$\frac{1}{\frac{1}{L} - \frac{1}{H}} = \frac{LH}{H - L}$$

DKO USD Calls with strike $1/H$ and DKO USD Puts with strike $1/L$, both with barriers $1/H$ (lower) and $1/L$ (higher). Since furthermore (in EUR terms)

$$\begin{aligned} & 1 \text{ DKO USD Call with strike } 1/H \text{ and barriers } 1/H \text{ and } 1/L \\ & = (1 \text{ DKO EUR Puts with strike } H \text{ and barriers } L \text{ and } H) / (HS_0), \end{aligned}$$

and similarly

$$\begin{aligned} & 1 \text{ DKO USD Put with strike } 1/L \text{ and barriers } 1/H \text{ and } 1/L \\ & = (1 \text{ DKO EUR Calls with strike } L \text{ and barriers } L \text{ and } H) / (LS_0), \end{aligned}$$

we obtain for the EUR-USD double-no-touch paying one unit of EUR (foreign currency)

$$DNT(L, H) = \frac{H \cdot DKOCall(L, L, H) + L \cdot DKOPut(H, L, H)}{(H - L)S_0},$$

where $DKOPut(H, L, H)$ means a EUR Put with strike H and barriers L and H and $DKOCall(L, L, H)$ means a EUR Call with strike L and barriers L and H . The division by EUR-USD Spot S_0 must be omitted, if the price of the DNT is to be quoted in EUR. If it is quoted in USD, then the formula stays as it is.

Static Replication of a FOR-Paying Double-No-Touch with One Double-Knock-Out Option

The static replication via spot inversion still requires two options. We now tackle the question how we can statically replicate a DNT paying one unit of foreign currency using just *one* double-knock-out option?

Solution

The payoff of a double-no-touch paying one unit of foreign currency is equivalent to ST domestic units (provided the spot stays in the range). Since

$$S_T = \max(S_T - 0, 0),$$

we can replicate the double-no-touch with a *zero-strike call* with the same barriers, notional and time to maturity, as shown in [Figure 2](#).

Perfect Replication: EUR-DNT with One DKO

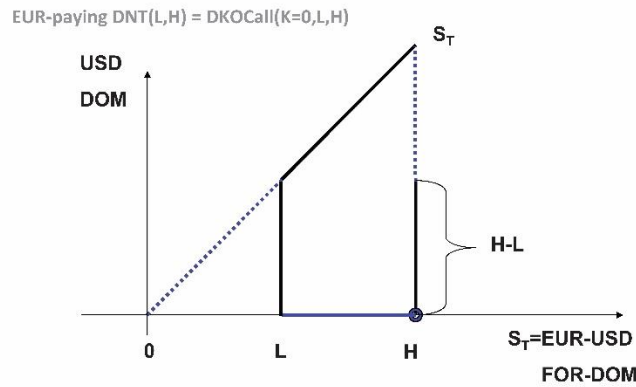


Figure 2: Payoff of a Zero-Strike DKO EUR Call with Barriers L and H

Reality Check

I know it. When we try to verify the theoretical result in a pricing system, we will find astonishing results. For example, in EIKON, I price the 6-month EUR-USD DNT paying 1M EUR with lower barrier 1.0500 and higher barrier 1.1500 on a spot reference of 1.1000 on 31 Dec 2023 and get a bid-offer of USD 356,465 – 386,370, see [Figure 3](#).

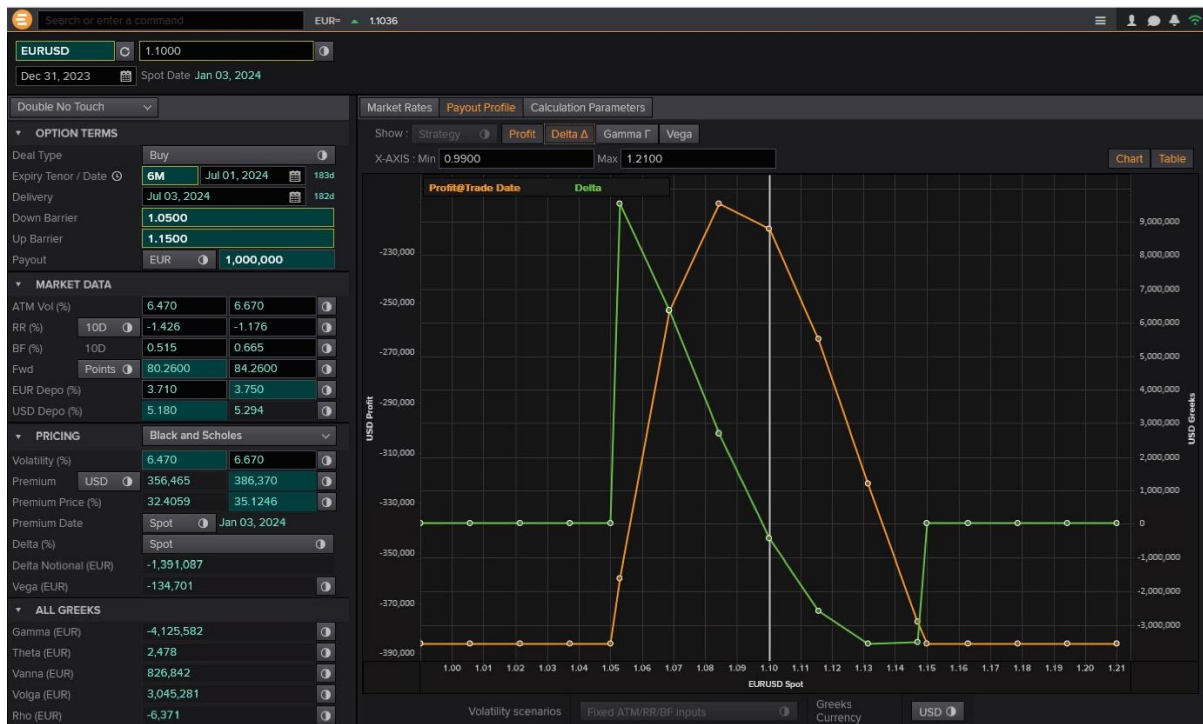


Figure 3: Pricing a 6M DNT in EUR in EIKON

Pricing the replication, a 6-months DKO EUR call on 1M EUR with strike 0.0001 (the system complains if I use 0.0000) lower barrier 1.0500 and higher barrier 1.1500 on a spot reference of 1.1000 on 31 Dec 2023 and get a bid-offer of USD 91,769 – 92,214, see Figure 4. I wish this was tradable. Both Prices are calculated in the Black-Scholes mode. We don't want to know what happens with other models. I encourage you to explore prices in other models on your favorite platform.

I suspect what went wrong in EIKON is that for the EUR call a volatility of around 58.5% has been calculated internally, using some extrapolation method for a very far away strike of zero. However, if the strike is zero, the volatility doesn't matter for the vanilla EUR call as the exercise probability is 100%. Volatility is relevant only for the barrier hitting probability. A proper comparison should be conducted with the same and a flat volatility.

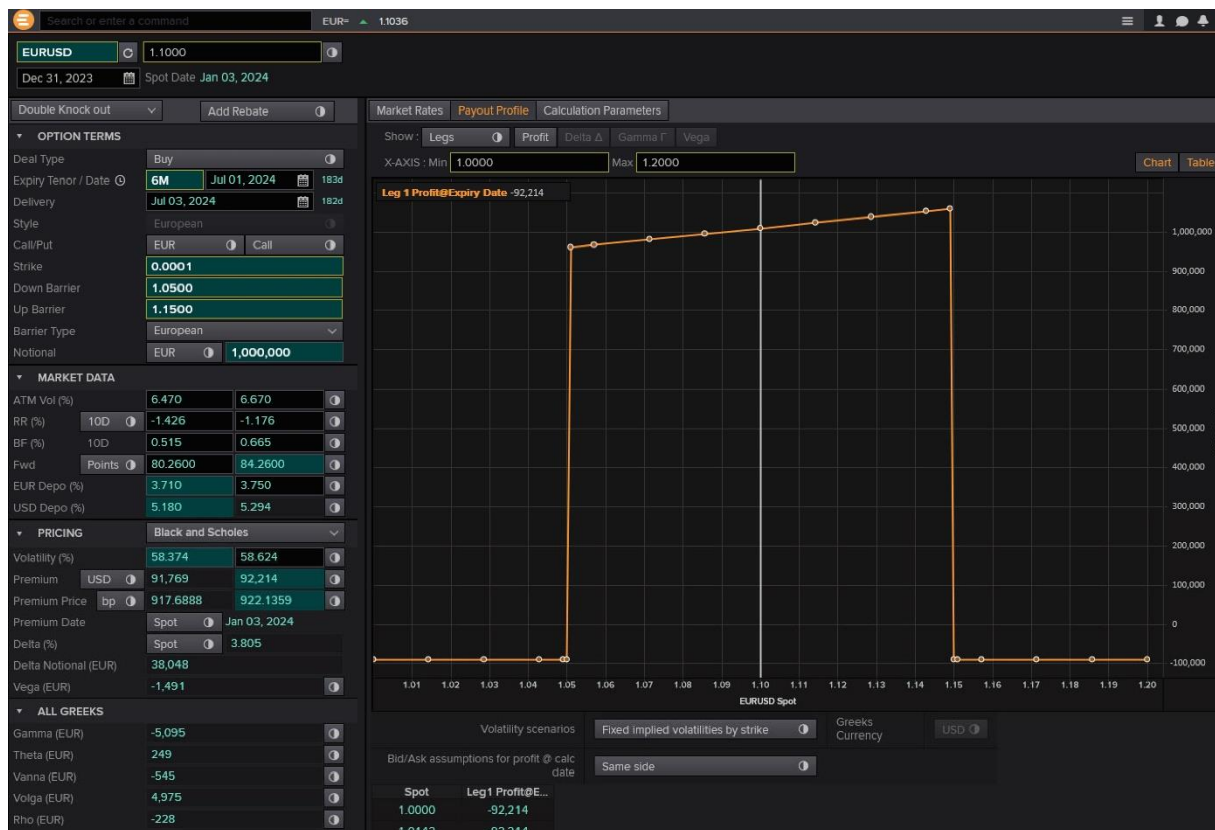


Figure 4: Pricing a EUR-USD 6M Zero-Strike EUR Call in EIKON

Conclusion

1. Replication examples for exotics are forever insightful.
2. A static replication of a EUR-paying DNT works with just one single EUR call, namely a zero-strike call.
3. The theoretical result may be applied in validation models in risk management systems or maybe used by traders in their search for arbitrage opportunities.