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FX Column: Quick and Dirty – Short Cuts for Option Lovers

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2023 is the 50th anniversary of the publication of "The Pricing of Options and Corporate Liabilities", i.e., the Black-Scholes model. I take the opportunity to highlight a few short cuts in option pricing, in the spirit of sanity checks, quick answers, passing interview questions.

Option Pricing 101: One of the first things I do when teaching FX options to a new group of delegates is ask the question what they think is the price on option – in percent of the notional amount. The obvious answers are typically of the level "it depends in the parameters". Indeed, knowing the Black-Scholes formula in the FX context in Figure 1, both market data spot S_0, forward $f_0(T)$, domestic and foreign interest rates r_d and r_f, volatility σ and contract data strike K, maturity t, delivery T, put/call indicator are all input and therefore, any value v(0) between 0% and 100% can occur.

$$
\begin{aligned}
v(0) &= e^{-r_d T}\left[f_0(T)N(d_+) - KN(d_-)\right] \\
&= S_0 e^{-r_f T} N(d_+) - K e^{-r_d T} N(d_-) \\
d_\pm &= \frac{\ln \frac{f_0(T)}{K} \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \\
&= \frac{\ln \frac{S_0}{K} + (r_d - r_f)T \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}
\end{aligned}
$$

Figure 1: Value of a Call Option in the Black-Scholes Model

In practice, when the sell-side trades with a corporate treasurer, we often observe the 1% Rule:

The price of an FX vanilla option is 1%.

This sounds too simple, but in fact isn't a bad indicator. And I will tell you why: if the price of the option were 10%, and the junior treasurer being very excited about this wonderful option he found would ask his boss to sign the check, would most likely be fired. It is just way too expensive, and therefore wouldn't trade. If on the other hand the treasurer would ask the bank for an option as a hedge for his FX exposure, one that has a price near zero, then the sell-side bank would more reasonably propose trading a forward or a zero-cost strategy like a risk reversal. Again, the option would not trade. Hence, very often, the price being indeed 1% (or near there, the geometric average between 0.1% and 10%), one tries to find the contract parameters or waits for a suitable spot to land at a price of 1%. That's why the price of an FX option is 1%. For this, we don't even need the Black-Scholes formula; but we do need it to back-out the implied parameters that lead to this price.

Asian Options: The short-cut price works like this:

> Divide the volatility by 1.7 and use this smaller volatility in the Black-Scholes formula.

This is based on the fact, that the geometric average of a log-normally distributed random variable is log-normally distributed. The continuous geometric average in the Black-Scholes model has variance $\sigma^2/3$. The volatility must therefore be divided by $\sqrt{3}\approx 1.7$. One then argues further that the traded arithmetic average option is similar to the more theoretical geometric average option, and that the traded discrete average option is similar to the more theoretical continuous average option. So, there are a few approximations, but the short cut works well, at least as a quick sanity check. The Black-Scholes formula can easily be used, while for most average options there are no or no simple formulas in closed form[1].

Compound Options: Recap: a compound option is an option that allow the holder to exercise a mother option and upon exercise receives a daughter option, which is usually a vanilla option. More generally, there can be multiple decision times with grand-mothers and grand-daughters in what we call an installment option. This adds more flexibility, which is worth about 1.2 times the vanilla option price.

> Compound options are worth 20% more than the corresponding vanilla option.

Even if this is not perfect, it is still helpful: When request to quote more precisely, I calculate the vanilla price, which maybe is 1%, then add 20% to this, land at 1.2%. Half of this is 0.6%, which is what I use as the strike of the mother option. Figure 2 shows an example of a standard 6M-6M compound in EUR. The vanilla price is 2.5 % EUR, with 20% on top, the short cut price would be 3% in total. We need split the two prices into half, so get 1.5% for the mother option and 1.5% for the daughter. I did one iteration and lowered the strike of the mother option to

[1] Details can be found in the section on Asian options in Wystup: FX Options and Structured Products, 2nd Edition, Wiley

1.45%. SuperDerivatives then shows a trading desk offer of 1.48%; with sales margin, we arrive at 1.5%. Bang on! What I am concealing of course, is that it took me 3 years to find this example.

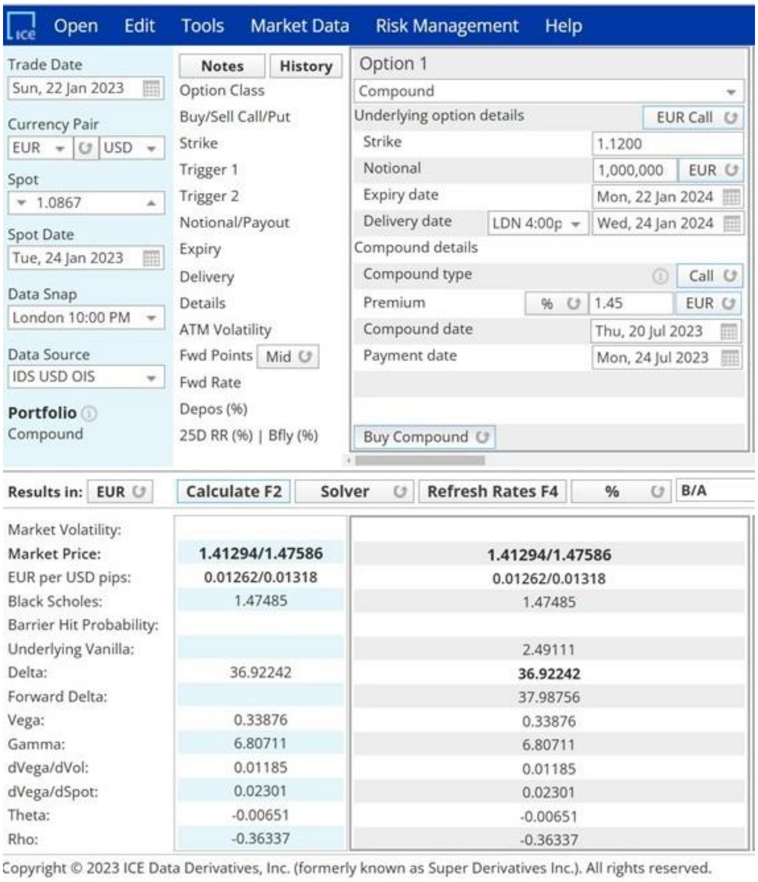

Figure 2: Pricing a Compound Option in SuperDerivatives

Regular Knock-Out Call: Pricing a regular knock-out call option via a semi-static hedge with a risk reversal is one of my favorite exercises in option training. The idea is to replicate a long regular knock-out call option with strike K above the spot (and with the barrier B below the spot) by a long vanilla call with the same strike K and a short vanilla put option with strike F^2/K, where F denotes the outright forward price at hitting time. If the spot does not hit the lower barrier until maturity, the put will be out of the money and the knock-out call will be a vanilla call. If the spot hits the barrier, then the knock-out call is knocked out and worth zero. The risk reversal should be closed at hitting time, so the bid price of the call should be equal to the offer price of the put. At hitting time spot will be on the barrier. We can then apply put-call symmetry to identify the strike price of the put. Call and put option have the same value in the Black-Scholes model if the geometric mean of the strike is equal to the forward. Assuming that the forward at hitting time is near the barrier (because there won't be much time left to maturity), we obtain the rule[2]

[2] Details can be found in Foreign Exchange Symmetries, *Contribution to* *Encyclopedia of Quantitative Finance*, John Wiley & Sons Ltd. Chichester, UK. 2010. pp.752-759.

Regular KO Call ≈ Vanilla Call – Vanilla Put struck at B^2/K

Digital Option Pricing: In FX, the digital call option pays a notional amount of a currency if the spot at maturity is at or above the spot (and zero otherwise). This is a path-independent payoff, but the value depends on the currency in which the notional is paid. We know or we can derive by staring at the Black-Scholes formula long enough that the following relationships hold (in fact exactly and independent of the model):

European Foreign Digital = Vanilla Delta

European Domestic Digital = European Foreign Digital

– Vanilla Value

One-Touch: In FX, the one-touch contract pays a notional amount of a currency if the spot trades at or above the continuously observed spot during the lifetime (and zero otherwise). It is similar to a European digital call, except that it is path dependent. If we denote by S the spot at maturity, and by M the maximum spot observed until maturity, we can write the payoff of a European digital with strike B with indicator functions as $1_{\{S \geq B\}}$ and the payoff of a one-touch with upper barrier B as $1_{\{M \geq B\}}$. These two payoffs are similar, and hence the question comes up how one can replicate a one-touch contract with European digitals and how they are mathematically related.

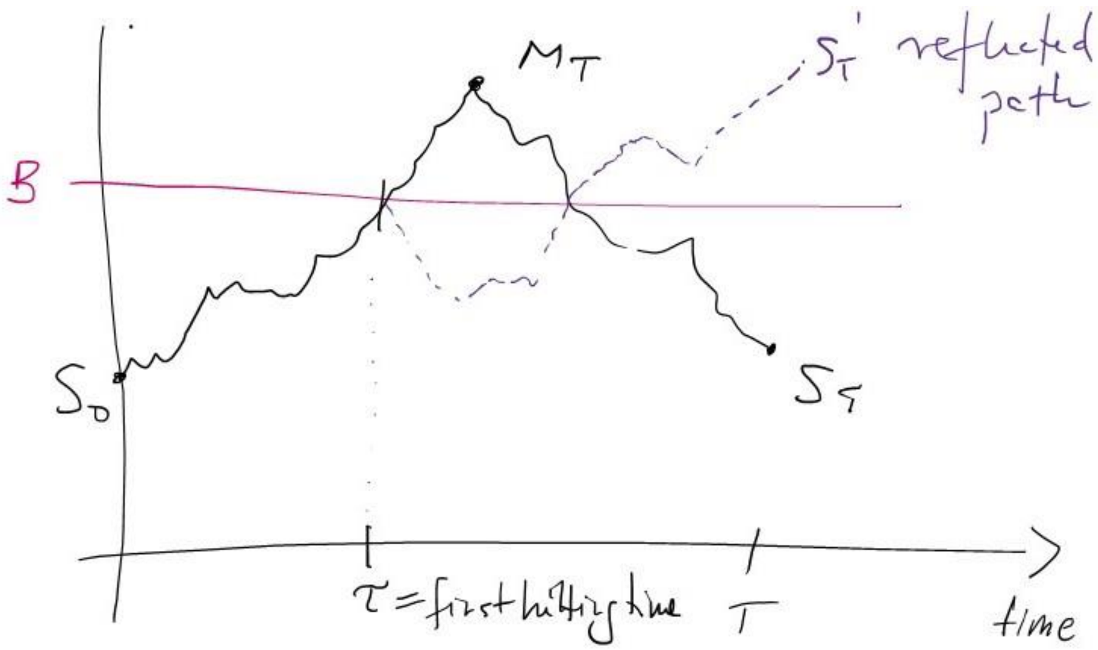

Figure 3: Reflection principle: For each path crossing the barrier B before maturity T and ending below B, there is a reflected path (dotted line) ending above B. In a symmetric diffusion model, the reflected path is equally likely.

The mathematics first: the reflection principle in Figure 3 illustrates a possible spot path starting at initial spot S_0, crossing the upper barrier B at first hitting time τ and landing at the final spot S_T at maturity T. If the reflected path starting at τ and ending above B is equally likely, we argue for the probabilities that

$$P\{M_T \geq B\} = P\{M_T \geq B, S_T \geq B\} + P\{M_T \geq B, S_T < B\}$$

$$= P\{S_T \geq B\} + P\{S_T \geq B\} = 2P\{S_T \geq B\}.$$

The probabilities are except the discount factor the value of the one-touch and the European digital. In a perfectly symmetric underlying the equations are exact. Translated to financial markets symmetry means a flat forward curve or equivalently zero swap points. If this is violated the equation above will only be an approximation. However, as a rule of thumb it tells us that a one-touch is roughly worth two European digitals.

One-Touch ≈ 2 European Digitals (Reflection Principle)

The semi-static replication: A trader can sell the one-touch and buy two European digital calls. If spot stays below the barrier B, everything will be out of the money and end with zero cash flows. If and when the spot crosses B, the trader would know at first hitting time τ that he needs to pay 100% of the notional of the one-touch at maturity. He would then sell the two European digital calls and receive two times 50%. The price of the European digital is not necessarily exactly 50% but depends on the forward and the windmill effect from the volatility smile[3].

[3] Uwe Wystup's FX Column "How Can a 50/50 Bet Have Odds of 1:2 Instead of 1:1?", Wilmott, Volume 2018, Issue 98, 14 November 2018, pp. 34-35

However, if the spot is at the barrier, the intuitive price of a digital is 50% of the notional. This semi-static replication argument confirms the above rule of thumb.

Basket Options: In a basket put option the holder has the right to exercise and upon exercise exchanges notionals of several currencies into a target currency. Alternatively, one could trade individual vanilla put options for each of the notionals. The correlation effect reduces the premium of the basket, by roughly 20%[4].

Basket put ≈ sum of corresponding vanilla puts minus 20%

This is a quite rough, indeed, but given that a client will most likely ask for a basket price about 500 times before he trades, answering the first 400 of these 500 queries without entering all the trade details in the risk management system, can save you substantial time.

Vega Hedging: A common interview question when applying for an option trader role, is: given you just sold a two-year at-the-money option, how many one-year at-the-money options do you need to buy to be vega-neutral?". The answer can be literally derived from the formula for a vanilla options vega in the Black-Scholes model:

$$Se^{-r_f(T)}\sqrt{t}N'(d_+)$$

The term d_+ is zero for at-the-money options, and the value of the normal density $N'(d_+)$ will be the same for both the one-year and the two-year call. Also, the options will have the same spot reference S. The essential difference is the time to maturity t. The vega ratio is therefore $\frac{\sqrt{2}}{\sqrt{1}} \approx 1.4$. The trader needs to buy about 1.4 times the notional of the two-year call.

Vega ratio ≈ square root of the ratio of times to maturity

Variance Swap: The par rate, the rate that makes the variance zero cost can be approximated by the average of the squares of the volatilities in the smile.

[4] Jürgen Hakala and Uwe Wystup: Making the most out of Multiple Currency Exposure: Protection with Basket Options, *The Euromoney Foreign Exchange and Treasury Management Handbook 2002*. Adrian Hornbrook.

Variance swap par rate ≈ average smile volatility^2

There will be more short-cuts, I am sure, and I will be happy to get more input from you. One more thing, to conclude: Infinity = 42 in the infinite sum of the formula for vanilla option values in the Black-Scholes-Merton model. Who would have thought.

Conclusion

1. The Black-Scholes model and the inherent symmetry yields several useful approximations.
2. Quick and dirty short-cut rules enhance and reflect our intuition.
3. Short-cuts help running a quick sanity check.

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