FX Column: Going Forward Step by Step

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Theory and Practice: At university, when we attend a course on continuous time finance, we usually deal with time as a continues process and interest paid is a function of the that time. Even volatilities and forward rate are scaled by time. With continuous time we learn that in a local volatility model one can match a given set of market prices of vanilla options exactly. And that is because one can derive the probability density of the future spot directly as the second order derivative of the call price function with respect to strike. That much for the theory, which I also love deriving in my training courses. Today's practice lesson shows that one needs to apply extra care when using this second order derivative further in the Dupire formula along with the time-derivative to compute local volatilities. Many practitioners know this problem, so it's time to share the following example to illustrate the source of the problem.

The FX Forward is a Step Function of the *Delivery* Date

The Forward Process

In practice, the FX Forward as function of the expiry time is a step function: it increases from one day to the next in one step, but not continuously during the day. Moreover, it is not directly a function of the *expiry time*, but of the corresponding *delivery day*. Indeed, an *FX Forward* contract does not have an expiry date, but only a delivery date. The expiry date and time is a contractual feature of an *FX Option*: the holder of an option has the right to decide at the expiry time if she wants to exercise the option, and in case of exercise, she triggers a cash-flow, which will settle in the account on the delivery date, which is usually two business days after the expiry date. For FX Forward contracts, such a discretionary decision is not needed and not intended, which is why it is not part of the FX Forward contract. However, we need the FX Forward rate to correctly price FX Options, where expiry dates do matter. And when we then consider the FX Forward rate as a function of the expiry date, we face larger jumps in the function FX Forward rate of expiry time around weekends and holidays.

Example

We use the market data snapshot in Table 1. The forward rates are listed as swap points, e.g., the figure 23.888 for the 3-months tenor means that the outright forward rate will be spot + swap points = 1.17776 + 0.0023888 = 1.18015.

	ATM	25 RR	10 RR	25 BF	10 BF	Spot/Forward	Rate
Tenor	BBG	BGN				1.17776	USD
1W	7.8800	0.0425	0.0750	0.1400	0.4100	1.705	0.1000
2W	7.9775	0.1650	0.2950	0.1550	0.4725	3.420	0.0888
1M	7.6425	0.2675	0.4650	0.1750	0.5200	9.030	0.0830
2M	8.3775	0.3600	0.6275	0.2075	0.6300	16.205	0.0820
3M	8.0825	0.4050	0.7325	0.2350	0.7225	23.888	0.0790
6M	7.4825	0.4425	0.8150	0.3275	1.0575	52.512	0.0690
1Y	7.3150	0.4950	0.9350	0.3900	1.3600	101.540	0.0540
2Y	7.5775	0.3700	0.6900	0.4150	1.4625	207.000	0.0354

Table 1: EUR-USD Market Data of 8 Sept 2020: source Bloomberg, using the Bloomberg Composite BGN.

FX Forward as a Function of the Expiry Time

First, we exhibit the FX Forward rate as a function of the expiry time in Figure 1 for expiry dates 200 to 250 days from horizon 8 Sept 2020. It is a step function with the common trading-week and weekend pattern.

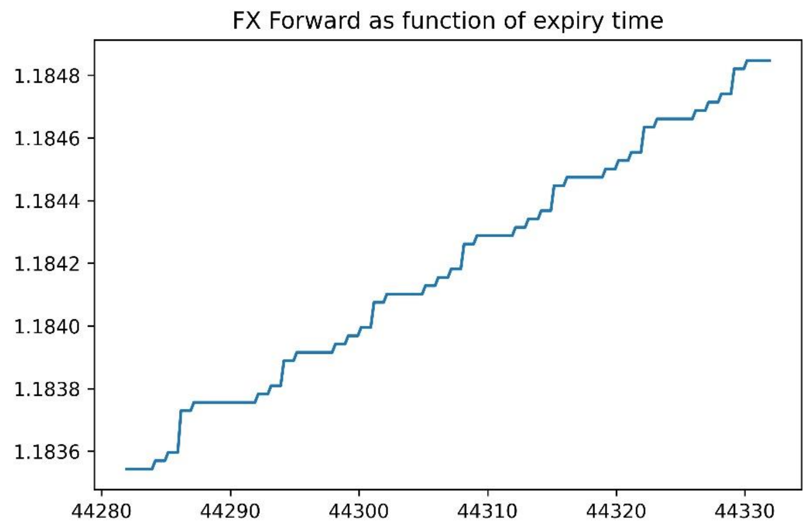

Figure 1: FX Forward as a Function of the Expiry Time. The x-Axis uses Dates in Excel Format. The y-Axis Units are USD per EUR

In textbook examples, and especially also in pricing algorithms for FX options the FX forward is often implemented as a continuous function of the expiry time, especially if the implementation applies numerical methods such as PDEs or Mont Carlo simulations.

Next, we compare FX Forwards implied from vanilla options in Figure 2. In the smooth version (Smooth) we assume that the FX forward rate was a continuous function of time. In the stepping version (Step), we use an implementation with the FX forward rate as a step function.

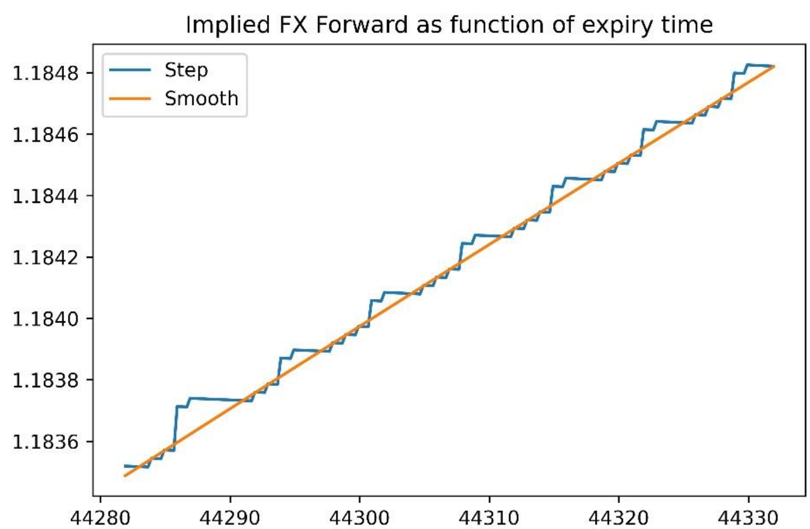

Figure 2: Implied FX Forwards as a Function of the Expiry Time: Continuous (labelled Step) vs. Step Function (labelled Smooth). The x-Axis uses Dates in Excel Format. The y-Axis Units are USD per EUR.

The difference can become an issue especially if time derivatives of options are computed by computing the differences after changing the time to expiry. We show the differences of the implied FX forwards to the FX forwards with an expiry 6 hours earlier in Figure 3. We clearly spot the pattern of 5-day trading weeks and weekends. Riddle: why is there a larger jump in the beginning?

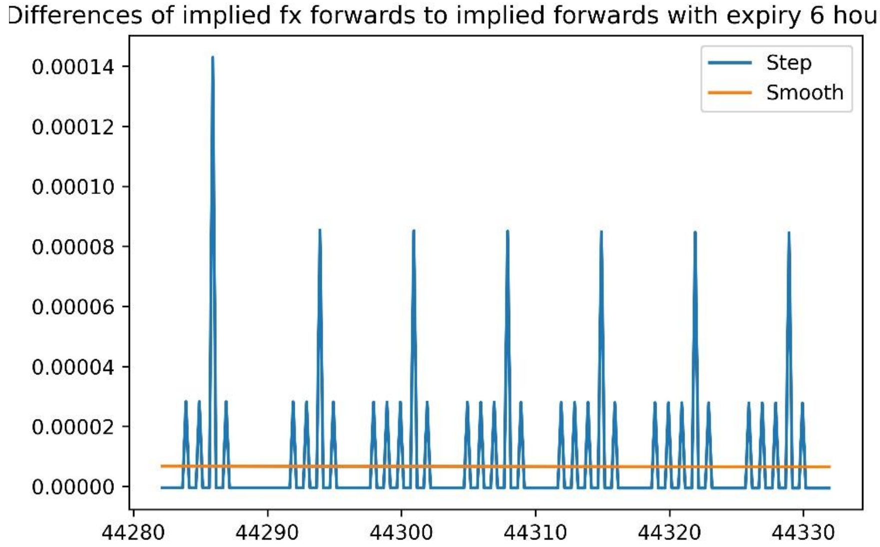

Figure 3: Differences of Implied FX Forwards to Implied FX Forwards with Expiry 6 Hours Earlier.

In the next two graphs we show this effect for the call option prices: In Figure 4 we compute call prices on a 6-hourly grid of the expiry time between 200 and 250 calendar days.

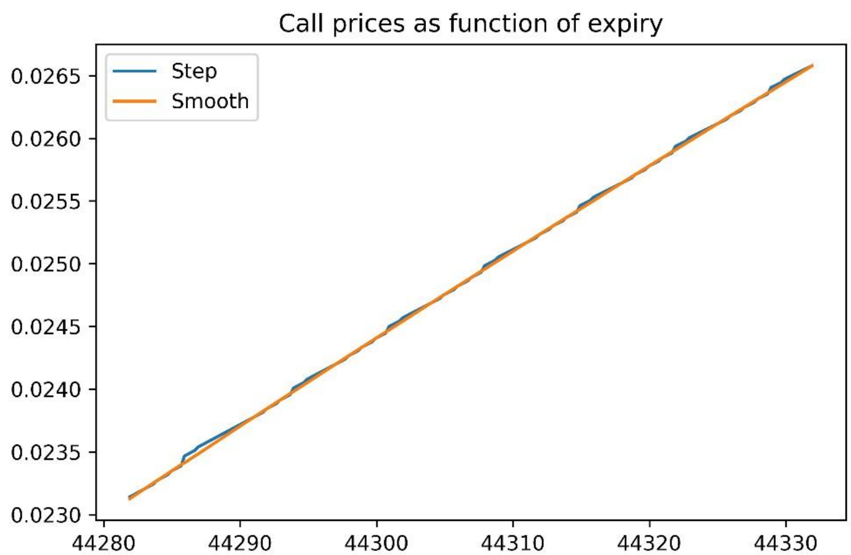

Figure 4: How Smooth vs. Step FX Forward Impacts the Vanilla Call Prices. The x-Axis uses Dates in Excel Format. The y-Axis Units are USD.

Not much of a difference after the third glass of wine. Therefore, we illustrate the effect showing the differences in Figure 5, where we rediscover the pattern of 5-day trading weeks and weekends (and the same riddle).

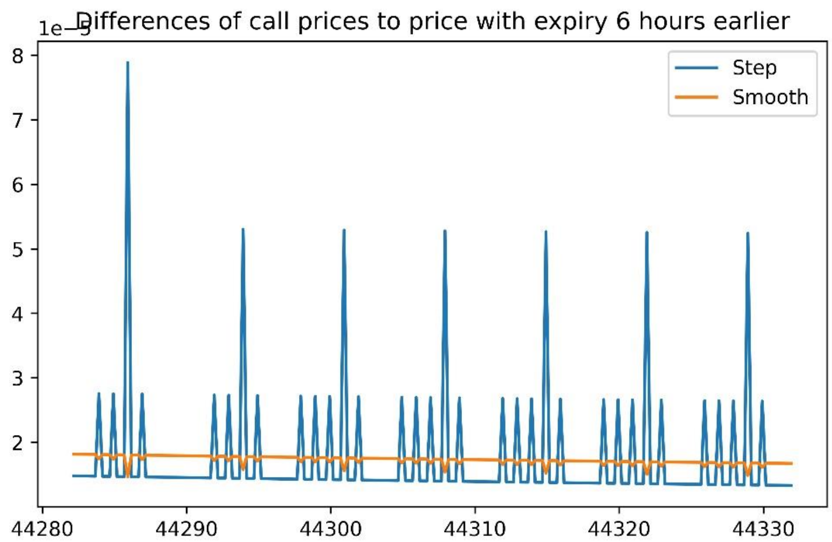

Figure 5: How Smooth vs. Step FX Forward Impacts the Vanilla Call Prices – Showing Differences.

It is particularly important to consider the exact form of the FX forwards implementation when computing time derivatives of option prices, for example for computation of a smooth local volatility surface. The Dupire formula which we use to calculate local volatilities requires a derivative with respect to the expiry time of the options.

Local Volatility

Now we illustrate the effect of computing naive time derivatives of call prices by moving expiry time and applying finite differences. If the pricing algorithm uses the continuous (smooth) version of the FX forward rate, we obtain a stable time derivative and hence local volatility surface. If the pricing algorithm was implemented using the correct FX forward step function, the time derivatives (and hence local volatilities) are not stable. The graphs show the local volatility on the strike-space at an expiry after 200 (Figure 6), 201 (Figure 7), and 202 (Figure 8) days.

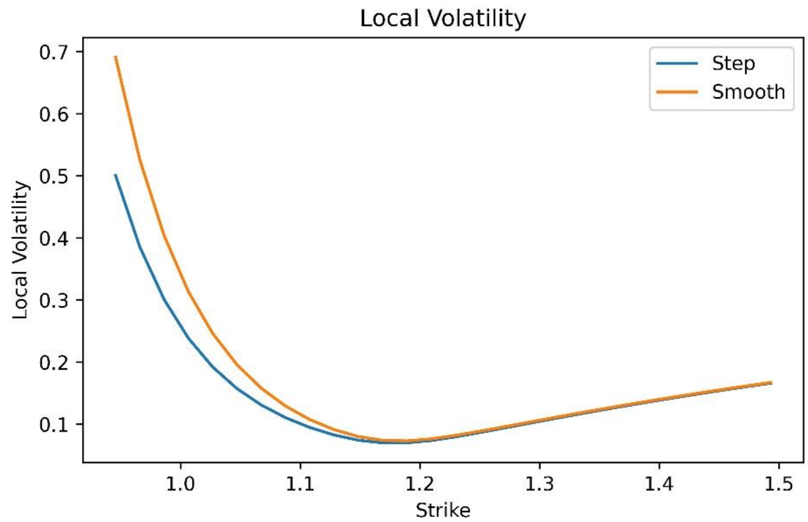

Figure 6: Local Volatility in Different Models with Expiry 200 Days.

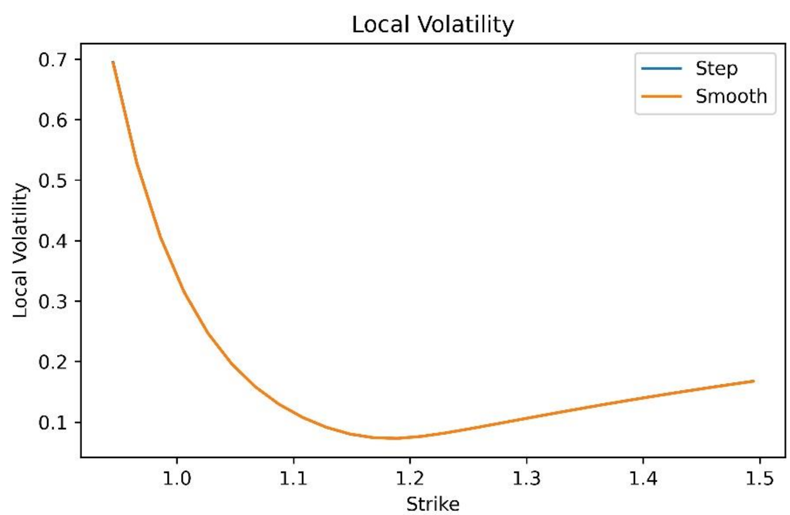

Figure 7: Local Volatility in Different Models with Expiry 201 Days.

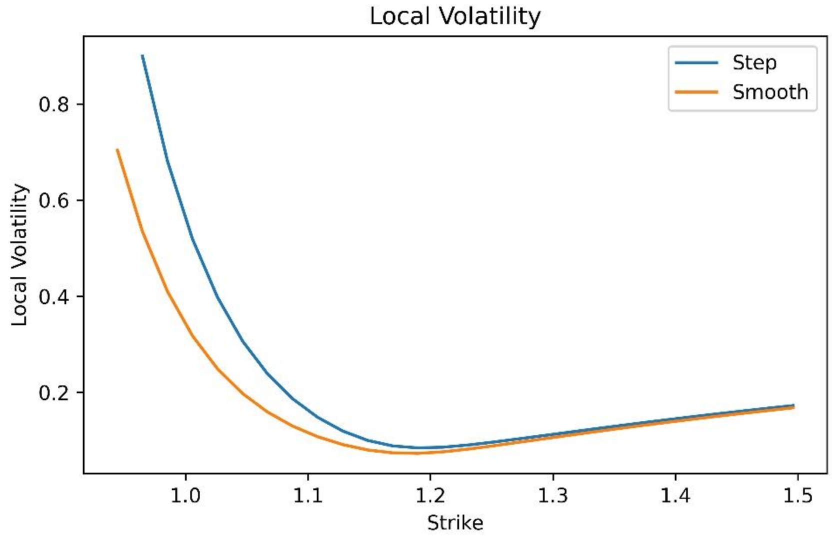

Figure 8: Local Volatility in Different Models with Expiry 202 Days.

We observe that the local volatility models only match the 201-days expiry but are below the volatilities in the Black-Scholes model for the 200-days expiry and above these volatility for the 202-days expiry.

Using Zero Rates and Zero Drift

To confirm the source of this effect, we remove the effect of interest rates and forward rates by setting the interest rates and the FX forward rate (the drift) to zero. In this simplified setup we illustrate in Figure 9 that in the step model we obtain the same local volatilities as smooth model.

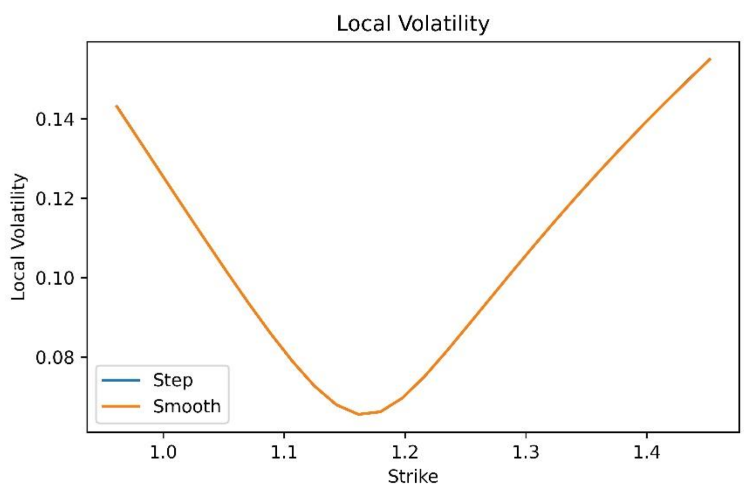

Figure 9: Local Volatility in Different Models with Expiry 202 Days and Zero Interest Rates and Zero Drift.

Conclusion

It is the step function pattern of FX forward rates that make naive time derivatives of call and put vanilla option prices appear volatile. When computing the Dupire local volatility with just this volatile time derivative and the second derivative with respect to strike, the estimate of the local volatility turns out to be also volatile. And this not caused by the structure in the volatilities, but just by not considering the term structure of the FX forwards appropriately.

As a result, one should either use the appropriate (in the sense of fitting the pricing algorithm's implementation) FX forwards that are used to backout implied volatilities and then compute the local volatilities from the implied ones. Or alternatively one can use the Dupire formula for call prices, but then needs to be precise when implementing the drift term.

Solution to the riddle: the larger jump in the beginning of Figure 5 is due to easter holidays in 2021.

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