FX Column: Can Butterfly be Negative and Why Do We Use the Probability Density to Check for Butterfly Arbitrage?

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Butterfly arbitrage means that a butterfly, a vanilla option based trading strategy with a non-negative payoff (Figure 1), has a negative price.

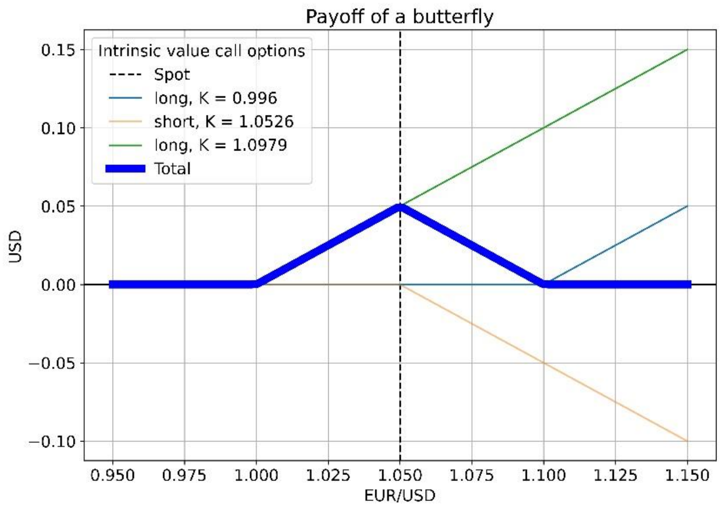

Figure 1: Payoff (in USD) of a butterfly: long call with lower strike, short 2 calls with middle strike, long call with higher strike

We usually check if a volatility curve for a fixed maturity is free of butterfly arbitrage by verifying that the implied probability density of the underlying future spot price is non-negative. Today, we recap

1. Can market quotes for butterflies be negative (smile)?
2. Can market quotes for butterflies be negative (broker)?
3. How are butterflies related to probability densities?
4. Why do we use probability densities to check for butterfly arbitrage?
5. If market quotes are negative, then how can a butterfly have a non-negative price?

The question of negative market quotes is easy to answer: yes, there are negative butterfly quotes. I know this because the delegates of my training courses on FX options have been spotting this for many years and were obviously curious, how the quotes can be negative, although we all learn that butterflies should not have negative price. As an example I show the 25-delta butterfly quotes in EUR-USD for the 2-week tenor for the time mid-June 2021 to mid-June 2022 in Figure 2. We clearly see that the quotes are mostly negative. So we don't need to dig very deep to find examples, and we don't need exotic currency pairs. The quotes are sources from Refinitiv Eikon and are labelled as brokers quotes for butterflies.

This means that a broker can price a strangle (the sum of a 25-delta call and a 25-delta put option) by adding the butterfly quote to the at-the-money (ATM) volatility and price the call and put with the same volatility, which will generate wrong prices for the call and the put, but a correct market price for the strangle. The butterfly can still have a positive price.

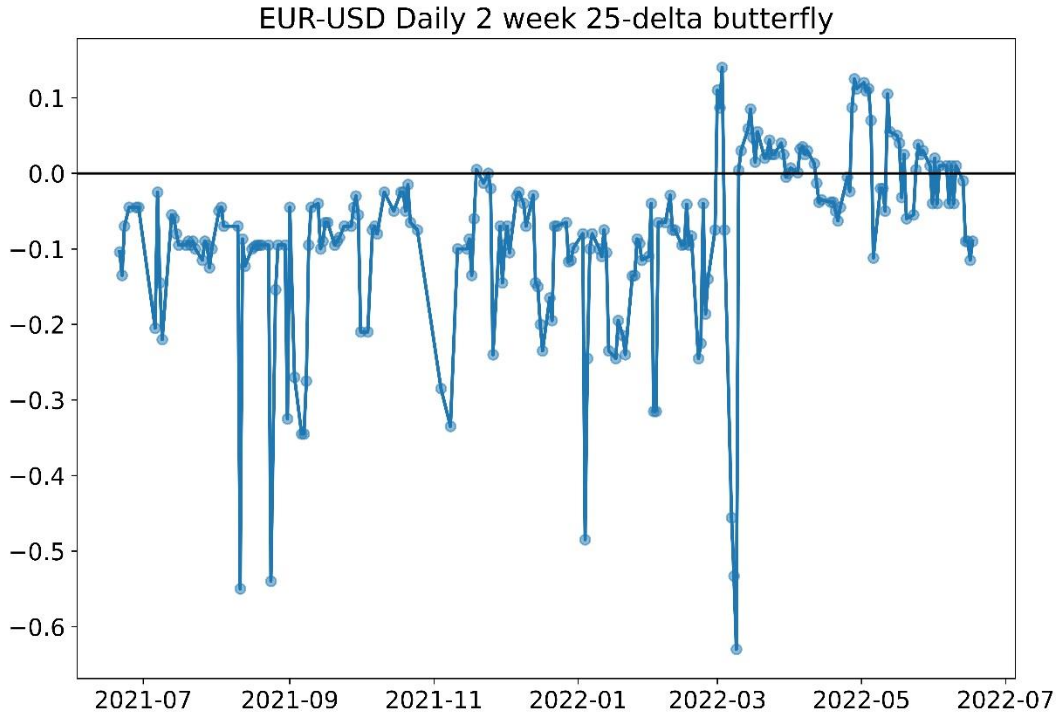

Figure 2: EUR-USD 2-week 25-delta butterfly bid quote in %, source via RIC EUR2WBF= in Eikon

Example

For example, let the spot price be 1.0500, we use zero interest rates, so forward is equal to spot, 10% ATM volatility, -2% Risk Reversal, -0.1% Butterfly. The 6-months (183 days) ATM strike is 1.0526, the 25-delta call volatility is 8.90% (8.92% in brokers convention), the 25-delta put volatility is 10.90% (10.92% in brokers convention). Assuming the 25-delta is attained for the correct volatilities in the smile curve, the strangle is worth 222 pips, The call with lower strike is 630 pips, the call with higher strike is 96 pips, the ATM call is 284 pips, thus the butterfly is 158 pips, substantially positive, summarized in in Table 1.

strike	strike (BF= -0.1%)	volatility	call value (USD pips)	Strike (BF= -3.0%)	volatility	call value (USD pips)
K_+	1.0979	10.92%	96	1.0817	8.03%	65
K_0	1.0526	10.00%	284	1.0526	10.00%	284
K_-	0.9996	8.92%	630	1.0121	6.03%	470

Table 1: Strike prices implied from brokers' quotation, corresponding volatilities and call option values

Hence, for a slightly negative butterfly quote of -0.1% (which is in terms of a *volatility* difference), the *value* of the butterfly is still positive. If we assume a butterfly quote of -3.0%, much more negative, keeping the rest of the data unchanged, we would arrive at a value of the butterfly of -33 pips. Therefore, butterfly quotes can be negative, and everything is still ok, but if they are strongly negative, then we run into butterfly arbitrage. We must not confuse the quotes in volatility differences with prices.

Visual Inspection

With the BF=-0.1%, the smile curve on the delta space will look concave, see Figure 3, unusual, I admit, when we are used to smiling smiles.

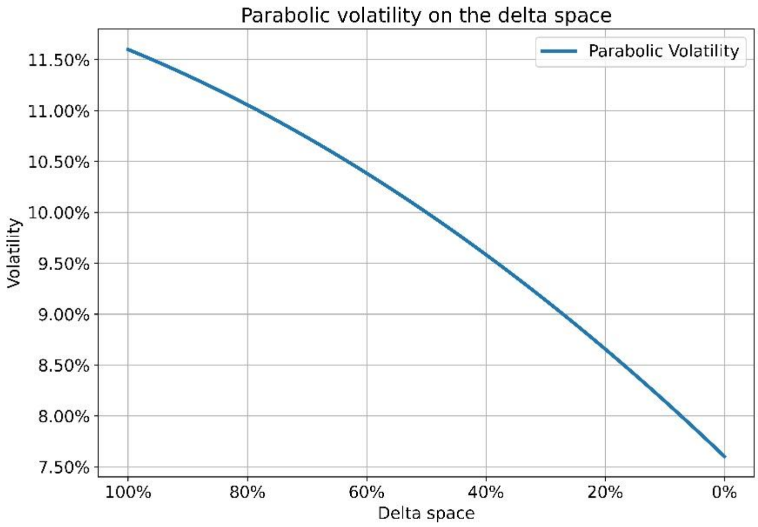

Figure 3: Implied volatility on the forward delta space (x-axis)

Once we convert the implied volatility to the strike space in Figure 4, it will start looking more familiar.

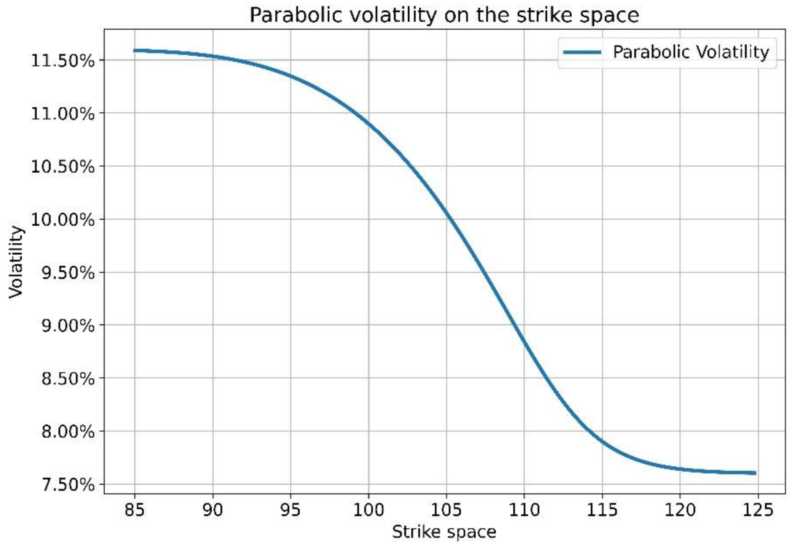

Figure 4: Implied volatility on the strike space (x-axis)

However, just looking at the smile curve does not necessarily tell us anything about butterfly arbitrage as explained in Wystup (2017). The real check is the probability density. The corresponding density in Figure 5 is non-negative and hence does not indicate a butterfly arbitrage.

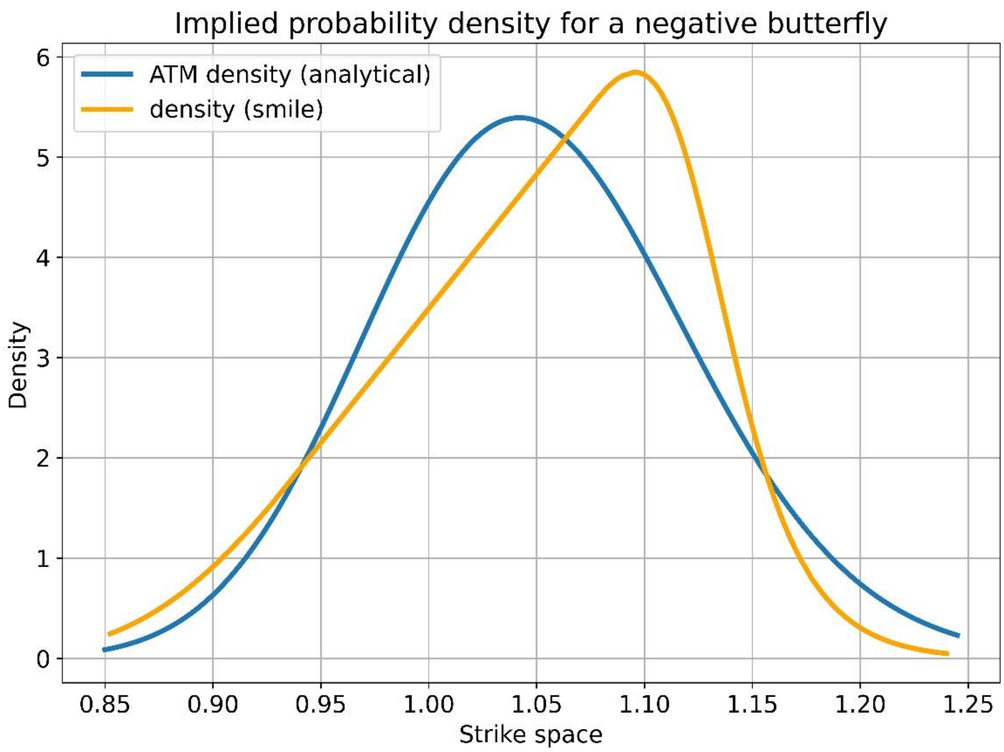

Figure 5: Implied probability density for a negative butterfly

How to Back Out the Density

This is an old hat, going back to a result by Breeden and Litzenberger (1978), but I like to tell the story anyway: Basic derivatives pricing theory tells us that the value of a call option as a function of the strike K is given by

$$c(K) = e^{-rT} E\left[\max(S_T - K, 0)\right]$$

$$= e^{-rT} \int_0^{\infty} \max(s - K, 0) p(s) ds,$$

where S_T denotes the spot price at maturity T, r is the domestic interest rate used for discounting, and p(s) is the probability density we want to know. Note that his call value function is a function of the strike, whose payoff is plotted in Figure 6. It looks like a put option payoff, but is actually a call option payoff, because is it plotted on the strike space with a frozen spot reference S.

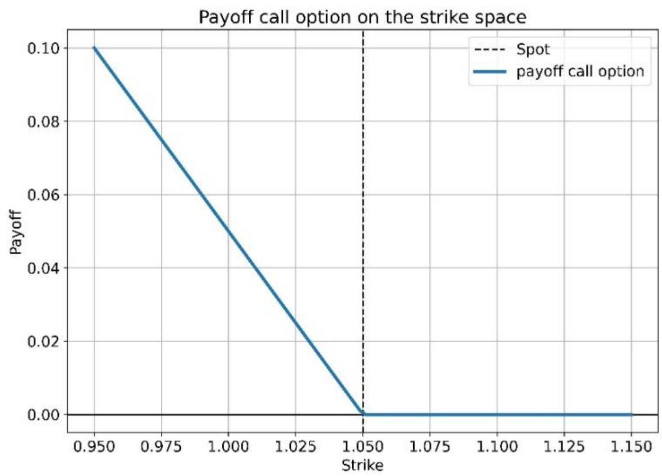

Figure 6: Call option payoff as a function of the strike

Differentiating twice with respect to K can be done using this plot. The positive part function differentiated once becomes an indicator function, and differentiated twice becomes a Dirac delta δ, a density that puts all the mass at K: The second derivative of the call value is therefore the (discounted) probability density.

$$c''(K) = e^{-rT} \int_{0}^{\infty} \delta(s - K) p(s) ds = e^{-rT} p(K)$$

As a consequence, once we know all the call option values, e.g. if we have successfully interpolated the volatilities on the strike space, the probability density is fully determined.

Relation Between Density and Butterfly

Numerically, we can approximate the probability density as the (compounded) second derivative of the call values on the strike space, i.e.

$$c''(K) \approx \frac{c(K-h) - 2c(K) + 2c(K+h)}{h^2}$$

for a small h. We recognize this finite difference approximation for the second derivative being the payoff of a butterfly (Figure 1). The butterfly is indeed a discrete version of the probability density. Now we see, how density and butterfly are related, and why we use the density to check for butterfly arbitrage.

Summary

1. Can market quotes for butterflies be negative (smile)? Yes, it happens frequently.
2. Can market quotes for butterflies be negative (broker)? Yes, it happens frequently.
3. How are butterflies related to probability densities? The butterfly is the discrete version of the density.
4. Why do we use probability densities to check for butterfly arbitrage? Checking the density means checking all continuous versions of butterflies at all strikes.
5. If market quotes are negative, then how can a butterfly have a non-negative price? Quotes are in volatility on the delta space, prices are in currencies. The units are different. But if butterfly quotes are strongly negative, then even the butterfly price can be negative.

References

1. Breeden, D.T. and Litzenberger, R.H. 1978. Prices of State Contingent Claims Implicit in Option Prices. Journal of Business, 51, 621-651.
2. Reiswich, D. and Wystup, U. FX Volatility Smile Construction, Wilmott, Volume 2012, Issue 60, pp 58-69.
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