Why Call Prices Decrease as the Probability of Up-Jumps Increase - A Riddle from the World of Jump Diffusion Processes

Uwe Wystup

uwe.wystup@mathfinance.com

MATHFINANCE

August 2021

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1 The Riddle

Recently, Sebastiaan Van Mulken, a student in my class on Monte Carlo Simulations and FX Derivatives Markets at the University of Antwerp approached me with the following question:

Why do the prices of a call option go down if we increase the probability of up–jumps?

First I asked him, whether he is sure he used a call option, rather than a put option instead and he responded with the graph in Figure 1, which shows that *both* the call *and* the put prices go down as the probability of an up-jump increases. Normally we would have expected that put prices decrease and call prices *increase*.

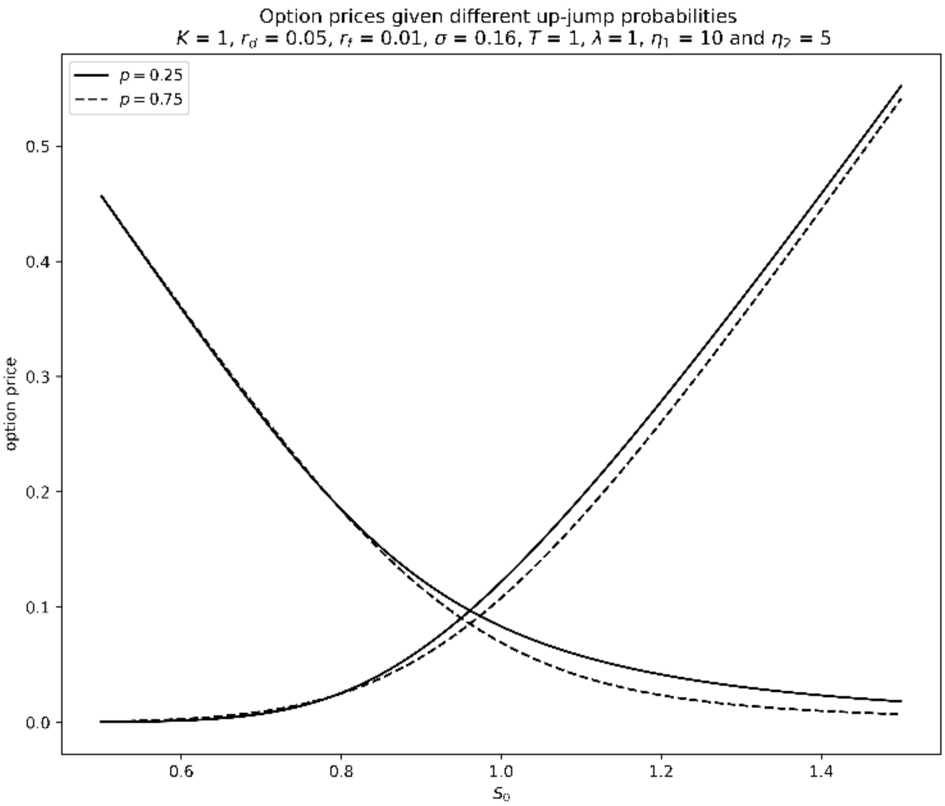

Figure 1: *Prices of call and put options in the Kou model as a function of the probability of up-jumps*

I started studying stochastic processes 30 years ago and am surprised, but also a bit happy I can still be surprised. In order to solve the riddle, let's recap the model first.

1.1 Kou's Double-Exponential Jump Diffusion Model

Following Kou [1] the double-exponential jump-diffusion process as a model for the spot price like an exchange rate of a currency pair FOR/DOM with foreign and domestic interest rates r_d and r_f is given by

$$\frac{\mathrm{d}S(t)}{S(t-)} = (r_d - r_f)dt + \sigma \mathrm{d}W(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \tag{1}$$

where $N(t)$ is a Poisson process with rate λ,
V_i i.i.d. non-negative random variables such that $Y = \ln V_i$ has an asymmetric double-exponential distribution with density

$$f_Y(y) = \begin{cases} p\eta_1 e^{-y\eta_1} & \text{if } y \geq 0, \\ (1-p)\eta_2 e^{y\eta_2} & \text{if } y < 0, \end{cases} \tag{2}$$

where $\eta_1 > 1$, $\eta_2 > 0$ and $0 \leq p \leq 1$, represents the probabilities of an upward jump.

$W(t)$, $N(t)$ and Y are all independent.
The solution to the SDE is

$$S(t) = S(0)\exp\left\{(r_d - r_f - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}\prod_{i=1}^{N(t)} V_i \tag{3}$$

To make the spot price a martingale we need to compute and add the drift adjustment

$$\nu \triangleq \mathbb{E}\left[e^Y - 1\right] \tag{4}$$

and can then implement a Monte Carlo simulation to compute the value of put and call options.

1.2 Call Option Price Formula

As a benchmark, on can compare simulation result with a transform-based value using the Lewis Method [2]. The call value version (3.11) of Lewis [2] is given by

$$C(S,K,T) = Se^{-r_f T} - \frac{1}{\pi}\sqrt{SK}e^{-(r_d+r_f)T/2}\int_0^\infty \Re\left[e^{iuk}\phi_T(u - \frac{i}{2})\right]\frac{\mathrm{d}u}{u^2 + \frac{1}{4}}, \tag{5}$$

$$k \triangleq \ln\left(\frac{S}{K}\right) + (r_d - r_f)T, \tag{6}$$

$$\phi_T(z) \triangleq \mathbb{E}\left[e^{izX_T}\right]$$

$$= \exp\left\{iz\omega T - \frac{1}{2}z^2\sigma^2 T + \lambda T\int_{-\infty}^{+\infty}(e^{izy} - 1)f_Y(y)\mathrm{d}y\right\}, \tag{7}$$

$$S(T) \triangleq S(0)\exp\{(r_d - r_f)T + X_T\}, \tag{8}$$

$$\phi_T(-i) = 1 \quad \text{martingale condition} \tag{9}$$

1.3 Properties of the Jump Factor

The parameter ω can be determined from the martingale condition and is (obviously) closely related to the drift-adjustment.
The expected value of the jump factor is given by

$$\mathbb{E}(V) = \mathbb{E}(e^Y) = q\frac{\eta_2}{\eta_2+1} + p\frac{\eta_1}{\eta_1-1}. \tag{10}$$

Furthermore, the expected value and variance of Y are given by

$$\mathbb{E}(Y) = \frac{p}{\eta_1} - \frac{q}{\eta_2}$$

,

$$\mathrm{Var}(Y) = pq\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^2 + \left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}\right).$$

2 The Solution

The riddles can be solved by computing the variance of the spot price, as call and put option prices are monotone functions of this variance. This requires the characteristic function.

2.1 The Characteristic Function and the Second Moment

Let's have a look at the Kou model's probability distribution of the spot. The idea is that from the characteristic function $\phi(t)$, we get the second moment $\mathbb{E}[X^2] = -\phi''(0)$, and from that we can calculate the variance as a function of the probability of an up-jump p, which I would expect to be decreasing. This will shed some light on the the riddle.
First of all, we observe that the expected value of Y goes up as the probability of an up-jump p goes up, while the expected value of $V = \exp Y$ goes down with an increasing probability p. We can simulate the spot prices using

$$S_T = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma B_t + B\cdot Y\right) = \underbrace{S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma B_t\right)}_{\text{Brownian}} \cdot \underbrace{\exp\left(B\cdot Y\right)}_{\text{jump factor}}. \tag{11}$$

Using the adjusted drift obtained from the martingale property $\phi_t(-i) = 1$ where $\phi(z)$ is the characteristic function corresponding to the double-exponential probability density, we obatin

$$\omega = -\frac{1}{2}\sigma^2 - \frac{\lambda p \eta_1}{\eta_1 - 1} - \frac{\lambda q \eta_2}{\eta_2 + 1} + \lambda. \tag{12}$$

The simulation step can hence be written as

$$S_{t+1} = \underbrace{S_t \exp\left((r_d - r_f + \omega)\delta_t + \sigma B_t\right)}_{\text{Brownian with adjusted drift}} \cdot \underbrace{\exp\left(B\cdot Y\right)}_{\text{jump factor}}. \tag{13}$$

2.2 Characteristic Function Calculations

Recall that if $f(x)$ is a probability density function, one defines the characteristic function as

$$\phi(z)_X = \mathbb{E}[e^{izX}] = \int_{-\infty}^{\infty} f(x)e^{itx}dx$$

Since $f(x)$ is a probability density function:

$$\phi(0) = \int_{\infty}^{\infty} f(x)\mathrm{d}x = 1.$$

If the random variable X has moments up to k-th order, then the characteristic function is k times continuously differentiable on the real axis and

$$\mathbb{E}[X^K] = i^{-k}\phi_X^{(k)}(0).$$

For the second moment one gets

$$\mathbb{E}[X^2] = -\phi_X^{''}(0). \tag{14}$$

Given

$$\phi_t(z) = \exp\left\{izwT - \frac{1}{2}z^2\sigma^2 T + \lambda T\left(\frac{p\eta_1}{\eta_1 - iz} + \frac{q\eta_2}{\eta_2 + iz} - 1\right)\right\} \tag{15}$$

one gets the first derivative

$$\begin{aligned}
\phi_t'(z) = \frac{\partial \phi_t}{\partial z} &= \left(i\omega T - z\sigma^2 T + \lambda T\left(\frac{ip\eta_1}{(\eta_1 - iz)^2} - \frac{iq\eta_2}{(\eta_2 + iz)^2}\right)\right) \\
&\quad \exp\left\{iz\omega T - \frac{1}{2}z^2\sigma^2 T + \lambda T\left(\frac{p\eta_1}{\eta_1 - iz} + \frac{q\eta_2}{\eta_2 + iz} - 1\right)\right\} \\
&= \left(i\omega T - z\sigma^2 T + \lambda T\left(\frac{ip\eta_1}{(\eta_1 - iz)^2} - \frac{iq\eta_2}{(\eta_2 + iz)^2}\right)\right)\phi_t(z),
\end{aligned} \tag{16}$$

leading to the first moment, i.e., the mean

$$\phi_t'(0) = i\left(\omega T + \lambda T\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)\right)\phi_t(0) \to \mathbb{E}[X] = i\phi_t'(0) = -\omega T - \lambda T\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right). \tag{17}$$

Here we used that $\phi_t(0) = \int_{-\infty}^{\infty} f_X \mathrm{d}x = 1$. The second derivative is given by

$$\begin{aligned}
\phi_t''(z) &= \frac{\partial}{\partial z}\left(i\omega T - z\sigma^2 T + \lambda T\left(\frac{ip\eta_1}{(\eta_1 - iz)^2} - \frac{iq\eta_2}{(\eta_2 + iz)^2}\right)\right)\phi_t(z) \\
&= i\omega T\phi_t'(z) - \sigma^2 T\phi_t(z) - z\sigma^2 T\phi_t'(z) + \lambda T\left(\frac{-2p\eta_1}{(\eta_1 - iz)^3} - \frac{2q\eta_2}{(\eta_2 + iz)^3}\right)\phi_t(z) \\
&\quad + \lambda T\left(\frac{ip\eta_1}{(\eta_1 - iz)^2} - \frac{iq\eta_2}{(\eta_2 + iz)^2}\right)\phi_t'(z).
\end{aligned} \tag{18}$$

Using $\mathbb{E}[X^2] = i^{-2}\phi_X^{''}(0) = -\phi_X^{''}(0)$ we further obtain

$$
\begin{aligned}
\mathbb{E}[X^2] &= -\left(\omega T i\phi_t'(0) - \sigma^2 T\phi_t(0) - 2\lambda T\left(\frac{p}{\eta_1^2}+\frac{q}{\eta_2^2}\right)\phi_t(0) + \lambda T\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right) i\phi_t'(0)\right) \\
&= -\omega T\left(-\omega T - \lambda T\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right) + \sigma^2 T + 2\lambda T\left(\frac{p}{\eta_1^2}+\frac{q}{\eta_2^2}\right) \\
&\quad -\lambda T\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\left(-\omega T - \lambda T\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right) \\
&= \omega^2 T^2 + 2\lambda\omega T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right) + \sigma^2 T + 2\lambda T\left(\frac{p}{\eta_1^2}+\frac{q}{\eta_2^2}\right) + \lambda^2 T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)^2. \quad (19)
\end{aligned}
$$

We can thus compute the variance via $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ and obtain

$$
\begin{aligned}
\text{Var}[X] &= \omega^2 T^2 + 2\lambda\omega T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right) + \sigma^2 T + 2\lambda T\left(\frac{p}{\eta_1^2}+\frac{q}{\eta_2^2}\right) + \lambda^2 T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)^2 \\
&\quad -\left(-\omega T - \lambda T\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right)^2 \\
&= \omega^2 T^2 + 2\lambda\omega T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right) + \sigma^2 T + 2\lambda T\left(\frac{p}{\eta_1^2}+\frac{q}{\eta_2^2}\right) + \lambda^2 T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)^2 \\
&\quad -\omega^2 T^2 - 2\lambda\omega T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right) - \lambda^2 T^2\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)^2 \\
&= \sigma^2 T + 2\lambda T\left(\frac{p}{\eta_1^2}+\frac{q}{\eta_2^2}\right). \quad (20)
\end{aligned}
$$

I have plotted the variance versus on the p-space in Figure 2, which shows that variance does indeed decrease with rising up-jump probability p. Since both, call and put prices are monotone functions of the volatility and hence the variance, we now understand that both call and put prices decrease with rising up-jump probability p.

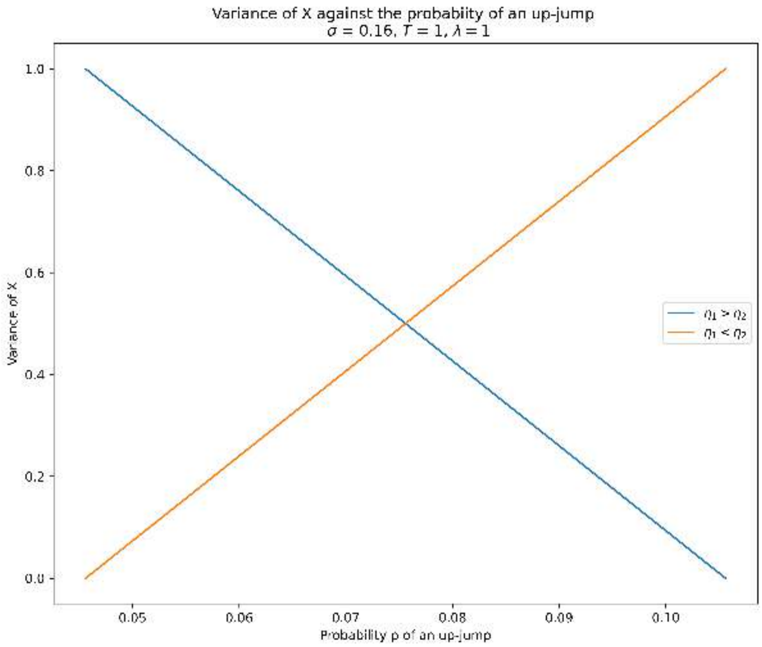

Figure 2: *Variance in the Kou model as a function of the probability of up-jumps when* $\eta_1 > \eta_2$

3 Conclusion

This jump-diffusion riddle is puzzling, but can be solved, and the good news is that in the course of preparing the solution, one can revisit many of the concepts of stochastic processes and probability theory, the calculations of transforms, the connection between transforms and moments, and generally have a real intellectual fun ride. However, note, the result is not as simple as it first appears, as everything of course depends on everything, so even business model of consulting ("it depends") can be applied.

The variance actually depends on the choice of η_1 and η_2, i.e., the sizes of the jumps. We have three cases (and remember, the size of the jumps is proportional to $1/\eta$):

1. $\eta_1 = \eta_2$: the jump up and down intensities are equal. In this case the variance will remain constant as p changes.

2. $\eta_1 > \eta_2$: the up jump is smaller than the down jump. In this case, the variance will decline as p increases as shown in Figure 2.

3. $\eta_1 < \eta_2$: the up jump has a higher intensity than the down jump. Here we see indeed what we expected, see Figure 3.

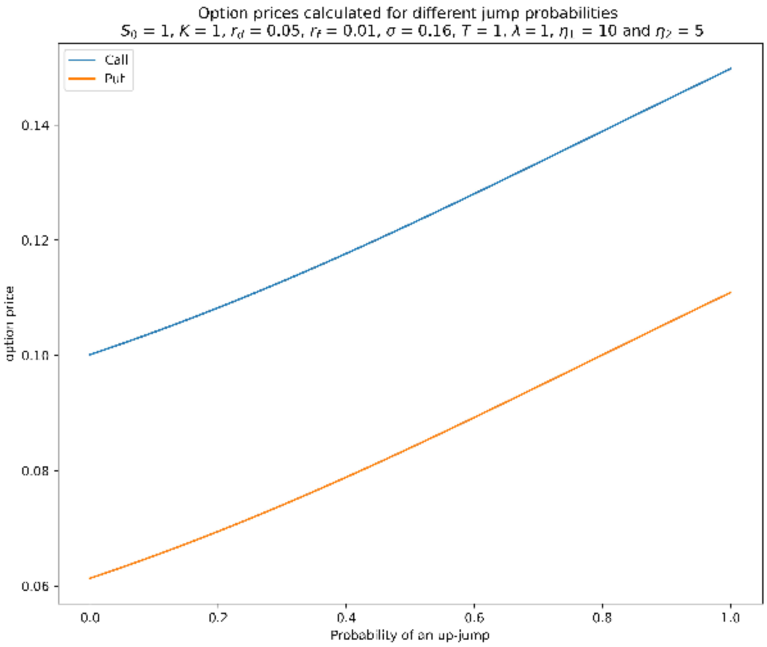

Figure 3: *Prices of Call and Put options in the Kou model as a function of the probability of up-jumps, when* $\eta_1 < \eta_2$

So the interesting case is when the jump down intensities are higher than the up ones, since even if you only have up jumps ($p \to 1$), the price of the call option still declines. To conclude I illustrate some scenarios of call and put prices on the p-space for various choices of model parameters in Figure 4.

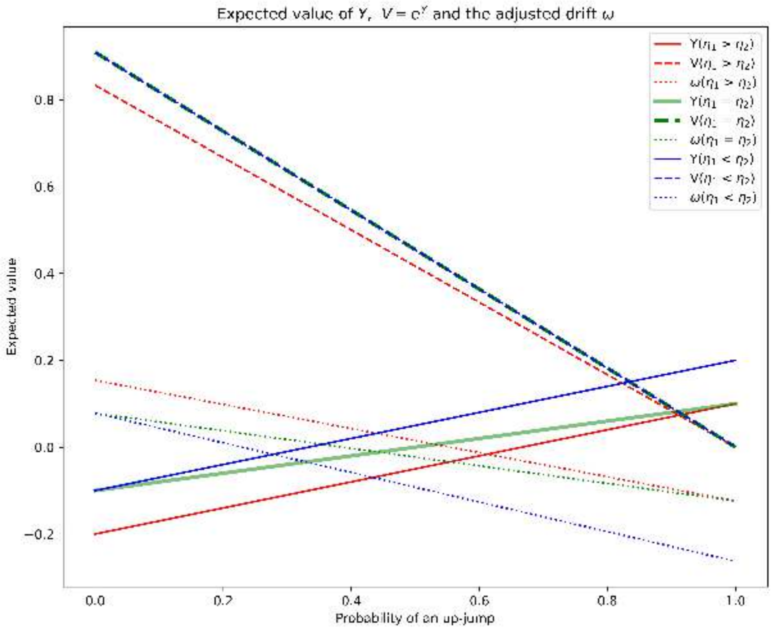

Figure 4: *Call and put option prices in the Kou model as a function of the probability of up-jumps for several parameter scenarios*

I am sure you get your hands on the transforms at your earliest opportunity to find more riddles and more solutions. I thank Sebastiaan Van Mulken for bringing this to my attention and helping me prepare this column.

References

[1] Kou, S.G. (2002). *A Jump Diffusion Model for Option Pricing*. Management Science 48, 8 (2002), pp. 1086 – 1101.

[2] Lewis, A. L. (2001). *A General Option Formula for General Jump-Diffusion and other Exponential Lévy Processes*. optioncity.net. Available at SSRN: http://dx.doi.org/10.2139/ssrn.282110