Valuation of Exotic Options under Shortselling Constraints as a Singular Stochastic Control Problem

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Abstract

This is a quantitative study of the valuation and hedging of *dangerous options*, options whose hedging strategies require unreasonable or risky short positions of the underlying instrument. We examine the valuation of many exotic options, when a shortselling constraint is imposed, as an example for *Contingent Claims in Incomplete Markets*. The valuation problem is known to be a stochastic control problem. We examine to what extend and under which conditions it can be viewed as a *singular stochastic control* problem.

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Contents

1 Preliminaries

The reader is advised to be familiar with stochastic calculus and Brownian motion.

1.1 Notation

Throuout we will use the notation

(0) $\triangleq$ to *define* a quantitiy,

(1) $a^+ \triangleq \max(a,0)$ for the positive part, $a^- \triangleq -\min(a,0)$ for the negative part,

(2) $a \wedge b \triangleq \min(a,b)$, $a \vee b \triangleq \max(a,b)$,

(3) $\mathbb{I}_B(x) \triangleq \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$ for the indicator,

(4) $f(t_-) \triangleq \lim_{s \uparrow t} f(s)$ for the left limit, $f(t_+) \triangleq \lim_{s \downarrow t} f(s)$ for the right limit,

(5) $v_x \triangleq \frac{\partial v}{\partial x}$ for the partial derivative of a function v,

(6) $\mathbb{E}[\dots]$ for the expectation, $\mathbb{E}^x[\dots] \triangleq \mathbb{E}[\dots | S(t) = x]$,

(7) $\mathbb{N}$ for the natural numbers, $\mathbb{Q}$ for the rational numbers, $\mathbb{R}$ for the real numbers,

(8) W for a Brownian motion, $\mathbb{P}$ for the Wiener measure on the set of continuous functions on the compact interval $[0,T]$,

(9) $\mathcal{L}$ for the Black-Scholes differential operator, i.e.

$$\mathcal{L}v(t,x) \triangleq v_t - rv + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx}$$

2 A motivating Example: The Up-and-Out Call

To Motivate a general theory on valuation of exotic options under shortselling constraints we treat a prototype of a dangerous path-dependent option: the up and out call, which can knock out in the money. This option causes hedging difficulties for the practitioner, because the hedger needs to take an unbounded position of the underlying instrument and this position is very unstable near the barrier. We impose a shortselling constraint of the hedging portfolio, discuss a possible analytical solution and show how to hedge. Finally we show that the valuation problem can be viewed as a *singular stochastic control problem.* This will be the basis for the general questions to be answered in the theoretical part.

2.1 The Hedging Problem

We model an underlying instrument, called the "stock" here, by a stochastic process $S(t)$. Stock can be anything like equity, commodity, currency, etc. Given the fair value of an option on the stock is $v(t,x)$, if the stock price is x at time t, then one usually hedges the option by replication of the payoff at the expiration time. It is done by holding $\Delta = v_x(t,x)$ shares of stock and investing $v(t,x) - xv_x(t,x)$ in the money market. In general this hedging portfolio must be updated instantaneously over time, and in a "complete market" [see e.g. KARATZAS and SHREVE] one can do this as well as holding unlimited positive or negative quantities of shares and cash. Of course, unrealistic assumptions can lead to unrealistic option values. For many options, particularly those, whose payoffs have kinks or (more dramatic:) jumps, the number $\Delta(t,x)$, the "delta" of the option, takes extremely large negative or positive values. For instance, an at-the-money digital call option one day before expiration, or more generally all types of barrier options, which can knock in or out in the money. In practice arbitralily large or small deltas are not acceptable: traders usually have a limited budget anyway, neither he nor any institution is willing to take the risk or the demanded amount of shares is just not available. What is the risk? Let us look at the problem of hedging a short up and out call with strike $K = 1.4000$ and barrier $B = 1.5000$. If the time to expiration $T - t = 1$day, then $\Delta(t, 1.4975) = -10$ is a realistic setup. Then if one is planning to adjust the delta every minute all the way up to expiration, one faces immense transaction costs, because as time changes, delta changes rapidly, even if the stock doesn't. If one decides to keep delta unadjusted over night or for a shorter time period, then two events can happen:

(a) if the stock stays below the barrier, say at 1.4950, then one had taken a large short postition in vain: one gains 250 pips due to a dropped stock price, but looses 950 pips, which is the obligation to the holder of the option. The total loss is 700 pips.

(b) if the stock moves up to 1.5000, then the option knocks out. To close out the position one needs to buy 10 shares at price 1.5000, so the total loss is 250 pips.

No matter what happens, the hedger faces guaranteed loss. In a complete market this risk is *not* reflected in the option value and therefore not in the hedge either. We will now illustrate how to find more realistic valuation and hedging procedures for such dangerous options.

2.2 Portfolio Constraints

To guarantee such extreme situations won't happen, one must impose portfolio constraints. A good number to control rather than just the delta is the *portfolio strategy*

$$\pi(t,x) \triangleq \frac{xv_x(t,x)}{v(t,x)},$$

which reflects the amount invested in the stock, measured in units of the option value. The portfolio strategy is sometimes called the *leverage* or the *gearing* of the option. The reason is: If the option is already practically worthless, one wouldn't worry about hedging it so much. On the other hand, the higher the option value, the more investment in the stock one would want to allow. Another advantage is the fact that $\pi(t,x)$ is dimensionless. Now it is our pleasure to review the results of [BROADIE, CVITANIĆ and SONER]. Assume the following standard model for the stock:

$$dS(t) = S(t)[rdt + \sigma dW(t)],$$

where σ is the volatility, r the risk free interest rate, $W(t)$ a Brownian motion under the risk neutral measure. Generally one tries to value an option given that $\pi(t,x)$ takes values in a closed, convex set C, for instance

(a) $C = [-\alpha, \infty)$ for some $\alpha \geq 0$ reflects a shortselling constraint or a shortselling prohibition for $\alpha = 0$.

(b) $C = (-\infty, \alpha]$ for some $\alpha \geq 0$ reflects a constraint to hold shares or a prohibition for $\alpha = 0$.

(c) $C = (-\infty, 1]$ reflects the prohibition to borrow from the money market.

(d) $C = (-\infty, \infty)$ reflects no constraints.

2.3 The Face-Lifting Equation

Define the *seller's cost* $v(0, S(0), C)$ of the option with payoff $\phi(S(T))$ to be the minimal initial amount of money (possibly infinite) which is needed to super-replicate $\phi(S(T))$ with a self-financing portfolio strategy $\pi(t,x)$, which satisfies $\pi(t,x) \in C$ for all $t \in [0,T]$ and all x. The *seller's cost* is also called *upper hedging price* [KARATZAS and SHREVE]. Super-replication means that we will have at least $\phi(S(T))$, possibly more, at expiration. We allow *super-replication* rather than exact replication, because obeying the portfolio constraint generally increases the final payoff following the principle: A final payoff with too many wrinkles has to be sent for face-lifting, before we compute its present value:

$$v(0, S(0), C) = e^{-rT} I\!E[\hat{\phi}(S(T))]$$

Here is the *face-lifting equation*:

$$\hat{\phi}(x) \triangleq \sup_{\nu \in \tilde{C}} \phi(xe^{-\nu}) e^{-\delta(\nu)},$$

where

$$\delta(\nu) \triangleq \sup_{\nu \in C} (-\pi\nu)$$

is the *support function* of the closed convex set C and

$$\tilde{C} \triangleq \{\nu : \delta(\nu) < \infty\}$$

is its *effective domain* [see ROCKAFELLAR]. The paper by [BROADIE, CVITANIĆ and SONER] proves this procedure in the multidimensional case, provides numerous examples for dimensions one and two and suggests a face-lifted payoff for lookback options. A general treatment of this procedure can be found in [KARATZAS and SHREVE], although there are no more explicit examples. To illuminate it for now, let us do the example of a non-path-dependent up and out call, which is of interest in our work anyway: Let

$$\phi(x) = (x-K)^+ \mathbb{I}_{[K,B]}(x)$$

for some strike K and barrier $B > K$. Impose the shortselling constraint $\pi(t,x) \in [-\alpha, \infty]$ for some $\alpha \geq 0$. Then $\delta(\nu) = \alpha\nu$ and $\tilde{C} = [0,\infty)$. A little ordinary calculus yields

$$\hat{\phi}(x) = \begin{cases} (x-K)^+ & \text{if } x \leq B, \\ (\frac{B}{x})^\alpha (B-K) & \text{if } x \geq B. \end{cases}$$

We could in fact compute the value function

$$v(t,x,C) = e^{-r(T-t)} \mathbb{E}[\hat{\phi}(S(T))|S(t) = x]$$

and the hedge $v_x(t,x,C)$ explicitely and observe that indeed the portfolio constraint holds. This constraint can be written in the form

$$\alpha v(t,x,C) + x v_x(t,x,C) \geq 0.$$

Before we proceed, let us understand the relation between this constraint and the face-lifting equation on an intuitive level: If we write the face-lifting equation as a real function of ν, namely

$$f(\nu) \stackrel{\Delta}{=} \phi(xe^{-\nu})\, e^{-\alpha\nu} \stackrel{!}{=} \max,$$

then the first order condition is

$$f'(\nu) = -e^{-\alpha\nu}[\alpha\phi(xe^{-\nu}) + xe^{-\nu}\phi'(xe^{-\nu})] \stackrel{!}{=} 0,$$

or in other words:

$$\alpha\phi(y) + y\phi'(y) \stackrel{!}{=} 0.$$

Since $\phi(y) = v(T,y)$, we see that the shortselling constraint is imposed with equality at the final boundary of the region where the Black-Scholes-equation is defined. One can check then that $v(t,x)$ satisfies the Black-Scholes-equation if and only if $\alpha v(t,x) + x v_x(t,x)$ does. It is now a consequence of the *maximum-principle* that the constraint holds inside this region as well, but not necessarily with equality. The reason why the shortselling constraint is imposed with equality at the final time is to get the minimality of the value function.
We will now impose this shortselling constraint on the path-dependent up and out call option. Before we solve it, we would like to give an interpretation from another point of view:

2.4 Interpretation of the Shortselling Constraint as a Transaction Cost Model

We consider the example of a European up and out call option on a stock, which has a payoff

$$(S(T)-K)^+ \mathbb{I}_{\{\max_{0\leq t\leq T} S(t) < B\}}$$

at the final time T. Here K is the strike price, B is the barrier and we assume $B > K$ throughout. As before let $v(t,x)$ be the value of the option at time t, when the stock price $S(t)$ is x. $v(t,x)$ can be computed explicitely, [see e.g. RICH]. It is also known that $v(t,x)$ is determined by the following:

$$\begin{aligned} -rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} &= 0 \\ v(t,B) &= 0 \\ v(T,x) &= (x-K)^+ I\!I_{\{x<B\}} \end{aligned}$$

The shortselling constraint we impose has been suggested first in [SHREVE], however, it had a quite different interpretation: The usual delta hedge suggests to hold $\Delta(t,x) \triangleq v_x(t,x)$ shares of stock at time t when the stock price is x. Since we want to hedge a short position, $\Delta(t,x)$ is negative near the barrier, and it turns out that it is unbounded near the barrier for t close to expiration as well. Thus it could be possible that the hedger would have to short sell a large amount of stocks. Since the gamma $v_{xx}(t,x)$ is also unbounded near the barrier, he would be in a very unstable situation: Either trading large amounts of stocks which causes substantial transaction costs. Or deciding to take only a bounded short position and do no further trading, in which case he would need some extra cash to close out his position if the stock moves up and crosses the barrier. To implement this in the model, S. Shreve has suggested to replace the boundary condition $v(t,B) = 0$ by $Bv_x(t,B) = -\alpha v(t,B)$. $Bv_x(t,B)$ is the dollar amount invested in the stock (a large negative number), if the stock price is near the barrier, and the value of the option should compensate losing a fraction $\frac{1}{\alpha}$ of this. Here the free nonnegative parameter α should reflect how much extra cash is needed: The less we need, the larger we must choose α. Only later we will prove that this α is actually the same as the shortselling parameter, which we have called α before as well. Now we are looking for a solution of the partial differential equation

$$\begin{aligned} -rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} &= 0, \quad 0 \le t < T, \quad 0 < x < B \\ \alpha v(t,B) + Bv_x(t,B) &= 0, \quad 0 \le t \le T \\ v(T,x) &= (x-K)^+, \quad 0 \le x \le B \end{aligned}$$

2.5 Explicit Solution for the Constrained Up-and-Out Call

To find the solution let

$$M(t) \triangleq \max_{0 \le u \le t} S(u).$$

We define the value of an auxiliary contingent claim by

$$w(t,x) \triangleq I\!E\left[e^{-r(T-t)}[(1+\alpha)S(T) - \alpha K] I\!I_{\{S(T)\ge K\}} I\!I_{\{M(T)<B\}} | S_t = x\right],$$

where we take the expectation under the risk neutral measure which makes $W(t)$ a Brownian motion. Now we can list some properties of $w(t,x)$:

(i) $e^{-rt}w(t,S(t))$ is a martingale, and therefore

(ii) $w(t,x)$ satisfies the Black-Scholes partial differential equation.

(iii) $w(t, B) = 0$

(iv) $0 \le w(t, x) \le (1 + \alpha)x$ and thus $w(t, 0) = 0$

(v) $w(T, x) = [(1 + \alpha)x - \alpha K]\mathbb{I}_{\{x \ge K\}}\mathbb{I}_{\{x < B\}}$

(vi) $w(t, x)$ is continuous on $[0, T] \times [0, B]$.

Now we can define

$$v(t, x; \alpha) \triangleq \int_0^1 y^{\alpha-1} w(t, xy) dy$$

and derive a list of properties of $v(t, x; \alpha)$:

(i) $v(t, x; \alpha)$ satisfies the Black-Scholes partial differential equation.

(ii) $0 \le v(t, x; \alpha) \le x$ and thus $v(t, 0; \alpha) = 0$.

(iii) $xv_x(t, x; \alpha) + \alpha v(t, x; \alpha) = w(t, x)$ and therefore in particular

(iv) $Bv_x(t, B; \alpha) + \alpha v(t, B; \alpha) = 0$.

(v) $v(T, x; \alpha) = (x - K)^+, \quad 0 \le x \le B$

(vi) $v(t, B; \alpha) = \int_0^1 y^{\alpha-1} w(t, By) dy$

(vii) $v(t, x; \alpha)$ is continuous on $[0, T] \times [0, B]$.

(viii) $\lim_{x \to 0} xv_x(t, x) = 0$

(ix) $v(t, x; \alpha) > v(t, x; \infty) \triangleq v(t, x)$ (follows from the maximum principle)

(x) $\lim_{\alpha \to \infty} v(t, x; \alpha) = v(t, x)$, as we will expect and see below.

Here we mean by $v(t, x)$ the value function of an up and out European call with the boundary condition $v(t, B) = 0$. In particular we learn that this $v(t, x; \alpha)$ does solve the problem: it superreplicates the payoff and satisfies the shortselling constraint. $v(t, x; \alpha)$ is the value function of an up and out European call option with a relaxed boundary condition at the barrier, and it has all the properties stated above. One would delta-hedge this option by holding $v_x(t, x; \alpha)$ shares of stock at time t, if the stock price at time t is x.

In the following, we will use the definition of $v(t, x; \alpha)$ to compute it explicitely. To do this, we need w first. Observe that $(M(t), S(t))$ is a Markov-process, and the payoff of w at expiration is a function of $(M(T), S(T))$. In addition, the value of $M(T)$ as such is not needed, we only need to know, whether it is below or above the barrier. We know that $w = 0$ in the latter case. In the other case it is sufficient to compute $w(0, x)$ as a function of T and afterwards replace T by $T - t$ to obtain $w(t, x)$. Fortunately the joint density of the random pair $(\max_{0 \le t \le T} W(t), W(T))$ is known for a standard Brownian motion $W(t)$ without drift. It is a straightforward application of the reflection principle. If we include a drift θ_- by setting $\tilde{W}(t) \triangleq W(t) + \theta_- t$ and $\tilde{M}(T) \triangleq \max_{0 \le t \le T} \tilde{W}(t)$, then we can do a change of measure to obtain the joint density for the pair $(\tilde{M}(T), \tilde{W}(T))$:

$$f(\tilde{m}, \tilde{w}) = \exp(\theta_- \tilde{w} - \frac{1}{2}\theta_-^2 T) \frac{2(2\tilde{m} - \tilde{w})}{T\sqrt{2\pi T}} \exp\left(-\frac{(2\tilde{m} - \tilde{w})^2}{2T}\right),$$

$$\tilde{m} > 0, \quad \tilde{w} < \tilde{m}, \quad \theta_\pm \triangleq \frac{r}{\sigma} \pm \frac{\sigma}{2}.$$

This is formula 1.1.8. of [BORODIN and SALMINEN 2.1]. Hence, we are able to compute the expected value as an integral:

$$\begin{aligned}
w(0, S_0) &= e^{-rT} \int_b^m \int_x^m [(1+\alpha)S_0 e^{\sigma x} - \alpha K] f(y,x)\, dy\, dx \\
&= (1+\alpha)S_0 \left[\mathcal{N}(\frac{m - \theta_+ T}{\sqrt{T}}) - \mathcal{N}(\frac{b - \theta_+ T}{\sqrt{T}}) \right] \\
&+ (1+\alpha)S_0 e^{2m\theta_+} \left[\mathcal{N}(\frac{m + \theta_+ T}{\sqrt{T}}) - \mathcal{N}(\frac{2m - b + \theta_+ T}{\sqrt{T}}) \right] \\
&- \alpha K e^{-rT} \left[\mathcal{N}(\frac{m - \theta_- T}{\sqrt{T}}) - \mathcal{N}(\frac{b - \theta_- T}{\sqrt{T}}) \right] \\
&- \alpha K e^{-rT} e^{2m\theta_-} \left[\mathcal{N}(\frac{m + \theta_- T}{\sqrt{T}}) - \mathcal{N}(\frac{2m - b + \theta_- T}{\sqrt{T}}) \right].
\end{aligned}$$

Here $\mathcal{N}(x) \triangleq \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ is the cumulative distribution function of a standard normal random variable and we abbreviate $m \triangleq \frac{1}{\sigma} \log \frac{B}{S_0}$ and $b \triangleq \frac{1}{\sigma} \log \frac{K}{S_0}$. Finally, it turns out that the integration needed to find v can be performed as well, and the result is

$$\begin{aligned}
v(t,x;\alpha) = x \Bigg[&\mathcal{N}\left(\frac{m}{\sqrt{\tau}} - \theta_+\sqrt{\tau}\right) - \mathcal{N}\left(\frac{b}{\sqrt{\tau}} - \theta_+\sqrt{\tau}\right) + e^{\frac{1}{2}s\tau(s-2\theta_+)} \\
&\left\{ e^{sm}\mathcal{N}\left(\frac{-m}{\sqrt{\tau}} + (\theta_+ - s)\sqrt{\tau}\right) - e^{sb}\mathcal{N}\left(\frac{-b}{\sqrt{\tau}} + (\theta_+ - s)\sqrt{\tau}\right) \right\} \Bigg] \\
+ x e^{2m\theta_+} \frac{s}{s - 2\theta_+} \Bigg[&\mathcal{N}\left(\frac{m}{\sqrt{\tau}} + \theta_+\sqrt{\tau}\right) - \mathcal{N}\left(\frac{l}{\sqrt{\tau}} + \theta_+\sqrt{\tau}\right) + e^{\frac{1}{2}s\tau(s-2\theta_+)} \\
&\left\{ e^{(s-2\theta_+)m}\mathcal{N}\left(\frac{-m}{\sqrt{\tau}} + (\theta_+ - s)\sqrt{\tau}\right) - e^{(s-2\theta_+)l}\mathcal{N}\left(\frac{-l}{\sqrt{\tau}} + (\theta_+ - s)\sqrt{\tau}\right) \right\} \Bigg] \\
- K e^{-r\tau} \Bigg[&\mathcal{N}\left(\frac{m}{\sqrt{\tau}} - \theta_-\sqrt{\tau}\right) - \mathcal{N}\left(\frac{b}{\sqrt{\tau}} - \theta_-\sqrt{\tau}\right) + e^{\frac{1}{2}\tilde{s}\tau(\tilde{s}-2\theta_-)} \\
&\left\{ e^{\tilde{s}m}\mathcal{N}\left(\frac{-m}{\sqrt{\tau}} + (\theta_- - \tilde{s})\sqrt{\tau}\right) - e^{\tilde{s}b}\mathcal{N}\left(\frac{-b}{\sqrt{\tau}} + (\theta_- - \tilde{s})\sqrt{\tau}\right) \right\} \Bigg] \\
- K e^{-r\tau} e^{2m\theta_-} \frac{\tilde{s}}{\tilde{s} - 2\theta_-} \Bigg[&\mathcal{N}\left(\frac{m}{\sqrt{\tau}} + \theta_-\sqrt{\tau}\right) - \mathcal{N}\left(\frac{l}{\sqrt{\tau}} + \theta_-\sqrt{\tau}\right) + e^{\frac{1}{2}\tilde{s}\tau(\tilde{s}-2\theta_-)} \\
&\left\{ e^{(\tilde{s}-2\theta_-)m}\mathcal{N}\left(\frac{-m}{\sqrt{\tau}} + (\theta_- - \tilde{s})\sqrt{\tau}\right) - e^{(\tilde{s}-2\theta_-)l}\mathcal{N}\left(\frac{-l}{\sqrt{\tau}} + (\theta_- - \tilde{s})\sqrt{\tau}\right) \right\} \Bigg]
\end{aligned}$$

Here we abbreviate $\tau \triangleq T - t$, $m \triangleq \frac{1}{\sigma} \log \frac{B}{x}$, $b \triangleq \frac{1}{\sigma} \log \frac{K}{x}$, $l \triangleq 2m - b$, $s = (1+\alpha)\sigma$, $\tilde{s} = \alpha\sigma$. Notice that in the second and in the fourth summand the denominator $s - 2\theta_+$ or $\tilde{s} - 2\theta_-$ could be zero for $\alpha = \frac{2r}{\sigma^2}$ or $\alpha = \frac{2r}{\sigma^2} - 1$ respectively. However, these are both removable discontinuities, and in fact one can apply l'Hôpital's rule to find the correct equation for these two points. Additionally we have so much freedom to choose α, that it is not worth putting another equation down, which is not more illuminating than the above.

From the equation it follows that $v(t,x;\alpha) = v(t,x) + u(t,x;\alpha)$, where the supplement $u(t,x;\alpha) > 0$ can be interpreted as a premium of an insurance for bounded leverage. (See below for a rebate interpretation of this supplement.) We can also derive that $\lim_{\alpha\to\infty} v(t,x;\alpha) = v(t,x)$.
To extend the formula to the case with dividends or foreign interest rates, replace r by $r_d - r_f$ in both θ_+ and θ_-, replace x by $xe^{-r_f\tau}$ and replace $e^{-r\tau}$ by $e^{-r_d\tau}$. Here, r_d stands for *domestic interest rate* and r_f stands for *foreign interest rate*, which could be a continuously paid dividend rate as well.

2.6 Comparative Statics

For practical use it seems handy to list some greek variables. Fortunately the auxiliary claim w helps us again:

Delta Solving the equation $xv_x(t,x;\alpha) + \alpha v(t,x;\alpha) = w(t,x;\alpha)$ yields

$$\Delta(t,x;\alpha) = \frac{w(t,x;\alpha) - \alpha v(t,x;\alpha)}{x}$$

immediately. We can use known results. The range for the relative delta at the barrier is clearly

$$\frac{Bv_x(t,B;\alpha)}{v(t,B;\alpha)} \in \{-\alpha\}.$$

Gamma Since $\Gamma(t,x;\alpha) = \frac{\partial}{\partial x}\Delta(t,x;\alpha)$, we can use the same trick again and get

$$\Gamma(t,x;\alpha) = \frac{xw_x(t,x;\alpha) - (1+\alpha)w(t,x;\alpha) + \alpha(1+\alpha)v(t,x;\alpha)}{x^2}$$

All we need is the delta for the auxiliary claim w. Here it is:

$$\begin{aligned}
w_x(t,x;\alpha) = (1+\alpha)&\left[\mathcal{N}(\frac{m-\theta_+\tau}{\sqrt{\tau}}) - \mathcal{N}(\frac{b-\theta_+\tau}{\sqrt{\tau}})\right] \\
-\frac{1+\alpha}{\sigma\sqrt{\tau}}&\left[\mathcal{N}'(\frac{m-\theta_+\tau}{\sqrt{\tau}}) - \mathcal{N}'(\frac{b-\theta_+\tau}{\sqrt{\tau}})\right] \\
-\frac{2r(1+\alpha)e^{2m\theta_+}}{\sigma^2}&\left[\mathcal{N}(\frac{m+\theta_+\tau}{\sqrt{\tau}}) - \mathcal{N}(\frac{2m-b+\theta_+\tau}{\sqrt{\tau}})\right] \\
-\frac{(1+\alpha)e^{2m\theta_+}}{\sigma\sqrt{\tau}}&\left[\mathcal{N}'(\frac{m+\theta_+\tau}{\sqrt{\tau}}) - \mathcal{N}'(\frac{2m-b+\theta_+\tau}{\sqrt{\tau}})\right] \\
+\frac{\alpha K e^{-r\tau}}{x\sigma\sqrt{\tau}}&\left[\mathcal{N}'(\frac{m-\theta_-\tau}{\sqrt{\tau}}) - \mathcal{N}'(\frac{b-\theta_-\tau}{\sqrt{\tau}})\right] \\
+\frac{2\alpha\theta_- K e^{-r\tau}e^{2m\theta_-}}{x\sigma}&\left[\mathcal{N}(\frac{m+\theta_-\tau}{\sqrt{\tau}}) - \mathcal{N}(\frac{2m-b+\theta_-\tau}{\sqrt{\tau}})\right] \\
+\frac{\alpha K e^{-r\tau}e^{2m\theta_-}}{x\sigma\sqrt{\tau}}&\left[\mathcal{N}'(\frac{m+\theta_-\tau}{\sqrt{\tau}}) - \mathcal{N}'(\frac{2m-b+\theta_-\tau}{\sqrt{\tau}})\right]
\end{aligned}$$

Here we abbreviate again $\tau \stackrel{\Delta}{=} T-t$, $m \stackrel{\Delta}{=} \frac{1}{\sigma}\log\frac{B}{x}$ and $b \stackrel{\Delta}{=} \frac{1}{\sigma}\log\frac{K}{x}$.
What is the range for gamma at the barrier? From above we get for $x = B$

$$v_{xx}(t,B;\alpha) = \frac{w_x(t,B;\alpha)}{B} + \alpha(1+\alpha)\frac{v(t,B;\alpha)}{B^2}.$$

Since the boundary condition $v(t, B; \alpha)$ has range $(0, B-K]$, we can obtain a relative gamma

$$\frac{B^2 v_{xx}(t, B; \alpha)}{v(t, B; \alpha)} = \alpha(1+\alpha) + \frac{B w_x(t, B; \alpha)}{v(t, B; \alpha)} \leq \alpha(1+\alpha),$$

because the delta of w must be negative at the barrier. We can also say that, as time approaches expiration, this delta must become arbitrarily small. We conclude that

$$\frac{B^2 v_{xx}(t, B; \alpha)}{v(t, B; \alpha)} \in (-\infty, \alpha(1+\alpha)].$$

Theta Using the Black-Scholes partial differential equation

$$v_t = r_d v - (r_d - r_f) x v_x - \frac{1}{2}\sigma^2 x^2 v_{xx}$$

and setting $\beta \triangleq \frac{\sigma^2}{2}(1+\alpha) - (r_d - r_f)$ implies

$$\Theta(t, x; \alpha) = -\frac{\sigma^2}{2} x w_x + \beta w + (r_d - \alpha\beta) v.$$

2.7 Minimality

Combining properties (iv) for w and (iii) for v, we conclude that v satisfies the portfolio constraint everywhere below the barrier and with equality at the barrier. We will now demonstrate that the function v derived above is the smallest function satisfying the portfolio constraint. To do this, we show that any other function $\tilde{v}(t, x; \alpha)$, which satisfies

- the Black-Scholes partial differential equation,
- $\tilde{v}(T, x; \alpha) = v(T, x; \alpha)$
- and the constraint $\alpha\tilde{v}(t, x; \alpha) + x\tilde{v}_x(t, x; \alpha) \geq 0$

can not be less than $v(t, x; \alpha)$. Since $\tilde{v}$ also satisfies the portfolio constraint at the barrier, but perhaps not with equality, let $\alpha\tilde{v}(t, B; \alpha) + B\tilde{v}_x(t, B; \alpha) \triangleq g(t) \geq 0$. Then $\tilde{v}$ can be characterized in the same way as v, namely by defining

$$\tilde{v}(t, x; \alpha) \triangleq \int_0^1 y^{\alpha-1} \tilde{w}(t, xy)\,dy$$

where

(i) $\tilde{w}(t, x; \alpha)$ satisfies the Black-Scholes partial differential equation.

(ii) $\tilde{w}(T, x) = w(T, x)$.

(iii) $\tilde{w}(t, 0) = w(t, 0) = 0$.

(iv) $\tilde{w}(t, B) = g(t) \geq 0 = w(t, B)$.

As before we conclude that

(i) $\tilde{v}(t, x; \alpha)$ satisfies the Black-Scholes partial differential equation.

(ii) $\tilde{v}(T, x) = v(T, x)$.

(iii) $\tilde{v}(t, 0) = v(t, 0) = 0$.

(iv) $\alpha\tilde{v}(t, x; \alpha) + x\tilde{v}_x(t, x; \alpha) = \tilde{w}(t, x)$ and hence

(v) $\alpha\tilde{v}(t, B; \alpha) + B\tilde{v}_x(t, B; \alpha) = \tilde{w}(t, B) = g(t)$.

Since by the maximum principle, $\tilde{w} \geq w$, we can deduce

$$\tilde{v}(t, x, \alpha) = \int_0^1 y^{\alpha-1}\tilde{w}(t, xy)dy \geq \int_0^1 y^{\alpha-1}w(t, xy)dy = v(t, x; \alpha).$$

Notice that $\tilde{w}$ can be viewed as an auxiliary up and out option with rebate $g(t)$, whereas w does not have a rebate. The option with the rebate must be worth at least as much as the option without the rebate. This is the maximum principle in terms of finance. In fact, if

$$\tau \triangleq \inf\{t : S(t) = B\}$$

then for $M(t) < B$

$$\begin{aligned}\tilde{w}(t, x) = I\!\!E \Big[& e^{-r(T-t)}[(1+\alpha)S(T) - \alpha K] I\!\!I_{\{S(T) \geq K\}} I\!\!I_{\{M(T) < B\}} \\ & + e^{-r(\tau - t)} g(\tau) I\!\!I_{\{\tau \leq T\}} \Big| S_t = x \Big]\end{aligned}$$

and $e^{-rt}\tilde{w}(t, S(t), M(t))$ is a martingale.
Result: The cheapest way to hedge without going too short in the stock *is* derived from the function v, even though one may not believe how much more expensive it is compared to the unconstrained option. The difference $v(t, x; \alpha) - v(t, x)$ describes the hedging difficulty quantitatively. It is a significant difference. As a trading institution one should not get scared at this point, because in our example we have isolated just one single option. In practice, however, one trades a whole book of options and one would then impose the shortselling constraint on the entire book. In this case the differences will usually be smaller. To get an idea, we will study a book of two up-and-out call options later in the examples.

2.8 $v(t, x; \alpha)$ as a Solution to a Singular Stochastic Control Problem

We set up the following stochastic control problem: Let us allow to push the geometric Brownian motion down in a possibly singular way:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) - S(t)d\lambda(t)$$

or equivalently for $t \geq u$

$$S(t) = S(u) \exp\left[\sigma(W(t) - W(u)) + (r - \frac{1}{2}\sigma^2)(t - u) - (\lambda(t) - \lambda(u))\right]$$

for an adapted nondecreasing control process λ starting at zero. We do not allow the control to push up, because the effective domain $\tilde{C} = [0, \infty)$, and we are looking for results in the same framework as in [KARATZAS and SHREVE]. To model the knock out feature, we introduce the function

$$h(x) \triangleq \begin{cases} 0 & \text{if } x \leq B, \\ -\infty & \text{if } x > B. \end{cases}$$

Now we can define $v(t, x; \alpha)$ as

$$\sup_{\lambda} I\!E \left[e^{-r(T-t)-\alpha(\lambda(T)-\lambda(t))} \phi(S(T)) + \int_t^T e^{-ru-\alpha\lambda(u)} h(S(u)) du \middle| S(t) = x \right]$$

Here ϕ depends only on the final value of the stock, e.g. $\phi(x) = (x-K)^+ I\!I_{\{x \leq B\}}$. The path-dependency of the payoff has been written using the function h. To interpret this, observe that λ should push the stock price down, if it gets above the barrier, because otherwise the function h will cause a negative infinite answer. But it should push with minimal effort, because it causes discounting. If λ pushes more than absolutely necessary, the supremum will not be achieved. We have called this supremum $v(t, x; \alpha)$. In the following we will establish that the explicit value function v defined by the list of properties above can in fact be written as the stated supremum. We are free to pose conditions for $x > B$. To make v and v_x continuous, we extend the boundary condition:

$$\alpha v(t, x; \alpha) + x v_x(t, x; \alpha) = 0, \quad 0 \leq t \leq T, \quad x \geq B.$$

We do not expect v_{xx} to be continuous at the barrier, but Itô's rule holds even without that. Now define

$$Y(t) \triangleq e^{-rt-\alpha\lambda(t)} v(t, S(t); \alpha) + \int_0^t e^{-ru-\alpha\lambda(u)} h(S(u)) du.$$

It will turn out that $Y(t)$ is a supermartingale for all controls λ and a martingale for the optimal control which we will call λ^*. We compute the differential:

$$dY(t) = e^{-rt-\alpha\lambda(t)}$$
$$\{-w(t, S(t)) d\lambda(t) + \sigma S(t) v_x(t, S(t); \alpha) dW(t) + \mathcal{L} v(t, S(t); \alpha) dt + h(S(t)) dt\}$$

$\mathcal{L}$ is the Black-Scholes differential operator:

$$\mathcal{L}(v(t, x)) \triangleq v_t - rv + rx v_x + \frac{1}{2}\sigma^2 x^2 v_{xx}$$

We will now integrate between t and T and take expectations conditioned on $S(t) = x$. The Itô-integrand $\sigma S(t) v_x(t, S(t); \alpha)$ takes values in $[-\alpha(B -$

$K), \sigma S(t)]$, whence its integral is a martingale.

$$\mathbb{E}^x[Y(T)] - \mathbb{E}^x[Y(t)]$$

$$= -e^{-rt-\alpha\lambda(t)}v(t,x;\alpha) + \mathbb{E}^x\left[e^{-rT-\alpha\lambda(T)}\phi(S(T)) + \int_t^T e^{-ru-\alpha\lambda(u)}h(S(u))du\right]$$

$$= -E^x\left[\int_t^T e^{-ru-\alpha\lambda(u)}w(u,S(u))d\lambda(u)\right]$$

$$+E^x\left[\int_t^T e^{-ru-\alpha\lambda(u)}\mathcal{L}v(u,S(u);\alpha)du\right] + E^x\left[\int_t^T e^{-ru-\alpha\lambda(u)}h(S(u))du\right]$$

$$\leq E^x\left[\int_t^T e^{-ru-\alpha\lambda(u)}(\mathcal{L}v(u,S(u);\alpha) + h(S(u)))du\right]$$

$$\leq E^x\left[\int_t^T e^{-ru-\alpha\lambda(u)}\mathcal{L}v(u,S(u);\alpha)\mathbb{I}_{\{S(u)=B\}}du\right] = 0.$$

We have used that w is nonnegative, $\mathcal{L}v + h = 0$ below B and $S(t)$ does not spend any time at the barrier. We conclude that for all control processes λ

$$v(t,x;\alpha) \geq \mathbb{E}^x\left[e^{-r(T-t)-\alpha(\lambda(T)-\lambda(t))}\phi(S(T)) + \int_t^T e^{-ru-\alpha\lambda(u)}h(S(u))du\right].$$

On reviewing the above computation, we realize that all inequality signs do actually become equal signs, if we choose the control

$$\lambda^*(t) \triangleq \sup_{0\leq u\leq t}(\log S(u) - \log B)^+$$

which only grows if $S(t)$ equals B. $\lambda^*(t)$ has singularly continuous paths, which gives our control problem its name. Recall that a real function f is called *singular*, if $f' = 0$ almost everywhere. For this optimal control process $\lambda^*(t)$, h is always zero and S is never above B. Finally, we replace the function h by the original path-dependent payoff and obtain

$$v(t,x;\alpha) \quad = \quad \sup_\lambda \mathbb{E}^x\left[e^{-r(T-t)-\alpha(\lambda(T)-\lambda(t))}[S(T)-K]^+\mathbb{I}_{\{\max_{0\leq u\leq T}S(u)\leq B\}}\right]$$

Remember that $S(t)$ depends on λ. We can also write down the same equation using the original $S(t)$:

$$v(t,x;\alpha) = \sup_\lambda \mathbb{E}^x\left[e^{-r(T-t)-\alpha(\lambda(T)-\lambda(t))}[S(T)e^{-\lambda(T)} - K]^+\mathbb{I}_{\{S(u)e^{-\lambda(u)}\leq B\forall u\}}\right]$$

Let us check the continuity of v_{xx} at the barrier: We have extended v above the barrier such that $\alpha v(t,x;\alpha) + xv_x(t,x;\alpha) = 0$ and therefore

$$v(t,x;\alpha) = v(t,B;\alpha)(\frac{B}{x})^\alpha.$$

It follows that

$$\begin{aligned} v_x(t,x;\alpha) &= -\frac{\alpha}{x} v(t,B;\alpha)(\frac{B}{x})^\alpha \\ v_{xx}(t,x;\alpha) &= \frac{\alpha(\alpha+1)}{x^2} v(t,B;\alpha)(\frac{B}{x})^\alpha. \end{aligned}$$

We read off a relative gamma

$$\frac{x^2 v_{xx}(t,x;\alpha)}{v(t,x;\alpha)} = \alpha(\alpha+1)$$

for all x above B and in particular as x goes down to B. However, we have noted before that the range of the relative gamma at the barrier is $(-\infty, \alpha(\alpha+1)]$. So indeed, v_{xx} is not continuous at the barrier for this choice of the extension. One may be warned that this is the reason, why we can not plug this v above the barrier into the Black-Scholes differential equation to derive properties of v_{xx} at the barrier.
To put this optimal control view into the framework of [BROADIE, CVITANIĆ and SONER], let us come back to the non-path-dependent up and out call. Its value function can be written as

$$v(t,x;\alpha) = \sup_\lambda \mathbb{E}^x \left[e^{-r(T-t)-\alpha(\lambda(T)-\lambda(t_-))} [S(T)-K]^+ \mathbb{I}_{\{S(T)\leq B\}} \right]$$

Here the maximizing control is

$$\lambda^*(t) \triangleq (\log S(T) - \log B)^+ \mathbb{I}_{\{t=T\}},$$

which is strikingly similar to the path-dependent one. We learn that in the non-path-dependent case, the control only acts at the final time, it pushes the stock down to the barrier if the stock is above the barrier and does not act otherwise. λ^* does not create any unnecessary discounting prior to the final time, because there is still a chance that the stock will end below the barrier, in which case it need not have pushed. It is time to point out here that we expect to find maximizing controls only in the set of adapted nondecreasing right-continuous processes starting at zero. Typically they are singular.

2.9 "Moving the barrier"

One of the most common answers of professionals to the question how they hedge a barrier option when the stock gets close to the barrier is "moving the barrier" up and do the valuation with this auxiliary barrier. We will now learn that this procedure agrees with imposing a shortselling constraint up to first order. In fact, our value function $v(t,x;\alpha)$ allows a justification of this custom and even a simple formula for the auxiliary barrier: We extend v above the barrier, such that it still satisfies the Black-Scholes differential equation. Then $v(t,x;\alpha)$ will be zero at some spot $x(t)$ for each t. $x(t)$ is the correct, but time dependent auxiliary barrier, which one should use, together with the auxiliary terminal condition $v(T,x;\alpha)$ for $B \leq x \leq x(T)$, to produce a value function for the original barrier option, which satisfies the constraint below the barrier. It does not satisfy the constraint above the barrier, but after the option knocks out, this doesn't matter anymore. $x(t)$ is time dependent, but it is "almost" constant over time. A very close *constant* approximation $x_0(t)$ can be obtained, if we extrapolate $v(t,x;\alpha)$ linearly above the barrier and choose $x_0(t)$ such that $0 = v(t, x_0(t); \alpha)$:

$$\begin{aligned} v(t,x_0(t);\alpha) &= v(t,B;\alpha) + v_x(t,B;\alpha)(x_0(t) - B) \\ &= v_x(t,B;\alpha)\left(-\frac{B}{\alpha} + x_0(t) - B\right) \end{aligned}$$

for $\alpha > 0$. In this case $v_x(t,B;\alpha) < 0$, which allows us to conclude that the curve $x_0(t)$, where $v(t,x_0(t),\alpha) = 0$ is a constant:

$$x_0(t) = B\frac{1+\alpha}{\alpha}$$

This is the auxiliary barrier, which one should use, together with a linear terminal condition, to produce a value function which satisfies the portfolio constraint approximatively. Finally observe that extending v above the barrier by requiring the constraint to hold with equality will yield $v(t,x;\alpha) = v(t,B;\alpha)(\frac{B}{x})^\alpha$, and in this case v will not be zero anywhere above the barrier.

2.10 Interpretation of $v(t, B; \alpha)$ as a rebate

In most of the finance literature barrier options usually have rebate features. Traded barrier options, however, are normally sold without any rebate agreements, mainly because options without rebate are cheaper than options with rebate, and secondly because a rebate is actually just a path-dependent digital option which can be separated easily from the barrier option and will be sold separately, if the need really occurs. For an out option a rebate agreement means that a sum R is paid from the seller of the option to the holder of the option, if the option knocks out. There are two kinds of agreements: (a) The rebate can be paid at expiry T, in which case the boundary condition of the Black-Scholes differential equation is $v(t,B) = Re^{-r(T-t)}$, or (b) the rebate can be paid at the first time τ the barrier is hit, in which case the corresponding boundary condition becomes $v(t,B) = R$. Both types can be viewed as an approximation to the function $v(t,B;\alpha)$. In any case, including such rebate features makes hedging easier, which could be one of the reasons as to why they were invented. The particular choice of the rebate $v(t,x;\alpha)$ is actually in favour of both the seller as well as the holder of the option: It is favourable for the seller, because it is exactly the kind of rebate one should specify in order to obey the portfolio constraint and for the holder, because his risk is small for times long before expiration and large for times near expiration.

2.11 List of Charts

We use these values for the parameters: $K = 1.4000, B = 1.5000, \sigma = 8\%, r_d = 5\%, r_f = 3\%, T = 50$ days, $\alpha = 50$.

(1) value of the unconstrained up and out call (without rebate)

(2) delta of the unconstrained up and out call (without rebate)

(3) gamma of the unconstrained up and out call (without rebate)

(4) theta of the unconstrained up and out call (without rebate)

(5) value of the unconstrained up and out call (with rebate paid at the end)

(6) value of the constrained up and out call

(7) delta of the constrained up and out call

(8) gamma of the constrained up and out call

(9) theta of the constrained up and out call

(10) difference of value functions of the constrained and the unconstrained up and out call

(11) difference of deltas of the constrained and the unconstrained up and out call

(12) value of the auxiliary up and out call

(13) delta of the auxiliary up and out call

(14) convergence of the constrained to the unconstrained call as α gets large

(15) relative gamma at the barrier

(16) rebate comparison and boundary condition $v(t, B; \alpha)$

(17) where to move the barrier

3 Overview of the Theoretical Results

3.1 Review of Existing Results

Pricing and Hedging of Contingent Claims in ideal complete and unconstraint markets has been understood very well. It is based on the fundamental principle of "absence of arbitrage opportunities." This price is called the *Black-Scholes* price [BLACK and SCHOLES, 1973], [MERTON, 1973]. We refer the reader to the article "On the pricing of Contingent Claims under Constraints (1996)" by Karatzas and Kou for a brief survey or the monograph "Methods of Mathematical Finance (1998)" by Karatzas and Shreve for a detailed discussion of the knowledge up to date. What we learn from there is that the price of a contingent claim is the unique one for which there are no arbitrage opportunities by taking either a short or a long position in the claim and investing wisely in the market. This price coincides with the minimal initial capital, starting with which one can exactly replicate the claim at the time of execution, and also with the expectation of the claim's discounted value under the unique, "risk-neutral" equivalent probability measure [HARRISON and PLISKA (1981)],[HARRISON and KREPS (1979)], [COX and ROSS (1976)]. The well-known argument leading to these results is essentially based on the martingale representation theorem and the Girsanov change of measure theorem from stochastic analysis [KARATZAS and SHREVE (1988)], [KARATZAS (1989)]. It has been pointed out that in the presence of *constraints* on portfolio choice there is no such unique price based solely on the principle of absence of arbitrage [KARATZAS and KOU, 1996]. One can only say that the unconstrained price lies between the so-called "lower hedging price" and "upper hedging price". The upper hedging price is the least price the seller can accept without the risk to violate the portfolio constraint. The lower hedging price is the greatest price the buyer can afford without the risk to violate the portfolio constraint. In the case of *convex constraints* on the portfolio choice, lower and upper hedging price can be characterized as a certain stochastic control problem, see additionally [CVITANIĆ and KARATZAS, 1993]. They point out that under appropriate conditions, it is possible to replicate contingent claims even with constrained portfolios, albeit some additional consumption may be necessary. [NAIK and UPPAL (1994)] first studied the effects of leverage constraints on the pricing and hedging of stock and bond options in discrete time. In the case of path-independent options [BROADIE, CVITANIĆ and SONER (1997)] extend these ideas and show that the upper hedging price can be found by pricing an appropriately increased dominating claim. We call this procedure *face-lifting*. [CVITANIĆ, PHAM and TOUZI (1997)] extend these ideas to a stochastic volatility model.

The main issues of this dissertation are a visualization of this stochastic control problem, an interpretation of this problem as a *singular* stochastic control problem and its application to exotic options. As an example for a constraint we put a shortselling constraint on the leverage and consider path-dependent European Contingent Claims. In a constant coefficient geometric Brownian motion model we are able to characterize many exotic options as solutions to certain partial differential equations (Lookback Put, Asian Put, Book of Barrier Options) or even compute prices analytically (Up and Out Call and Put Options). Our theory will clarify some of the relevant issues of lower and upper semicontinuity of the contingent claims, which have so far been neglected in the literature, and it will illuminate, *why* the wonderful results of [BROADIE, CVITANIĆ and SONER] are right: Allowing control processes to look like they typically are, namely singularly right-continuous, sheds more light on their ideas.

3.2 The New Discoveries

We work on the probability space $(\Omega, \mathcal{F}, I\!P)$, where $\Omega = \mathbf{C}[0,T]$ is the space of continuous functions on the interval $[0,T]$ and $I\!P$ is the Wiener measure. We let $\mathcal{F}_t$ be the (augmented) *Brownian filtration* and $\{W(t)\}_{0 \le t \le T}$ be a Brownian motion, adapted to this filtration. Our basis is a constant coefficient geometric Brownian motion as a model for the underlying instrument, called the"stock":

$$dS(t) = S(t)[rdt + \sigma dW(t)].$$

r is the risk-free rate, $\sigma > 0$ the volatility of the stock. Note that for instructional reasons we started with a stock, whose mean rate of return is *already* equal to the risk-free rate r. We will consider contingent claims $F \ge 0$ of the form $F = g(S(\cdot)) = g(\{S(t)\}_{0 \le t \le T})$. To find the value of F at time zero, one usually comes up with an F-replicating strategy $(\pi(t, S_t), C(t))$, where $\pi(t,x)$ is the number of shares of stock to hold at time t, if the stock price is x, and $C(t)$ is the consumption up to time t. This generates the portfolio value process $V(t)$

following the evolution

$$\begin{aligned} dV(t) &= \pi(t, S_t)dS_t + r(V(t) - \pi(t, S_t)S_t)dt \\ &= rV(t)dt + \sigma\pi(t, S_t)S_t dW(t) \end{aligned}$$

In a complete market the consumption process is zero. We consider our market incomplete in the sense that a shortselling constraint on the gearing, namely

$$\frac{S_t \pi(t, S_t)}{V(t)} \geq -\alpha, \ \alpha \geq 0, \tag{1}$$

is imposed. Exact F-repliction may now not work anymore, because it could potentially violate the shortselling constraint. Thus we admit a nondecrasing right-continuous aggregate consumption process $C(t)$ staring at zero. Alternatively we don't consume and then talk about F-*superreplication.* We want to clarify in this thesis what is the *upper hedging price* of F subject to the shortselling constraint. Our final vision is to view the constrained valuation problem of F as a *nonconstrained* valuation problem of a face-lifted dominating $\hat{F}$, similar to the procedure for path-independent options provided by [BROADIE, CVITANIĆ and SONER]. To get a definition of this $\hat{F}$, we need a maximizing control process λ^* and would then define

$$\hat{F} \triangleq e^{-\alpha\lambda^*(T)} g\left(S e^{-\lambda^*}\right). \tag{2}$$

In order to be able to state this definition we need to look for maximizing control processes in a sufficiently large class. We believe that this class is the collection of all adapted nondecreasing right-continuous processes starting at zero. Our starting point is the following *Main Hedging Result* by [CVITANIĆ and KARATZAS, Hedging Contingent Claims, Theorem 6.4.], later presented in [KARATZAS and KOU, Pricing Contigent Claims, Theorem 6.1] and in [KARATZAS and SHREVE: Methods of Mathematical Finance, Theorem 5.6.2]:

Theorem 3.1 *The upper hedging price is*

$$\sup_{\lambda} \mathbb{E}_\lambda \left[e^{-rT - \alpha\lambda(T)} g\left(S_0 e^{\sigma W_\lambda + \mu - \lambda}\right)\right],$$

where we maximize over all adapted, nondecreasing absolutely continuous control processes λ starting at zero, whose derivatives are uniformly bounded.

The Brownian motion W_λ has been defined to be

$$W_\lambda(t) \triangleq W(t) + \frac{1}{\sigma}\lambda(t),$$

and the expectation is to be taken under the probability measure $\mathbb{P}_\lambda$ defined as

$$\mathbb{P}_\lambda[A] \triangleq \int_A Z_\lambda(T)\, d\mathbb{P} \quad \forall\ A \in \mathcal{F}_T,$$

where the density process $Z_\lambda(t)$ is

$$Z_\lambda(t) \triangleq \exp\left\{-\frac{1}{\sigma}\int_0^t \lambda'(s)\, dW(s) - \frac{1}{2\sigma^2}\int_0^t (\lambda'(s))^2\, ds\right\}.$$

One of the problems is that this formulation of the main hedging result really stretches our imagination: We are maximizing over changes of measure, a rather non-transparent procedure. Our new new version of the main hedging result basically says that we do not need the subscript λ at $I\!\!E$ and W, i.e. we will show that the upper hedging price can be written as

$$\sup_{\lambda} I\!\!E\left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right].$$

We can now view the constraint valuation problem as a maximization over processes with absolutely continuous nondecreasing paths that have bounded derivatives and conveniently forget that a change of measure has ever taken place. The control problem has been visualized! The fact that *all* our examples show maximizing control processes that have *singular* nondecreasing paths motivates us to go on and show that we really only need to maximize over continuous controls:

Theorem 3.2 *If F is lower semicontinuous, then the upper hedging price is*

$$\sup_{\lambda} I\!\!E\left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right],$$

where we maximize over all adapted, nondecreasing (possibly singularly) continuous control processes λ starting at zero.

The lower semicontinuity of F is not a necessary condition, but if F is not lower semicontinuous, the theorem may fail at some starting points S_0. We prove this theorem by constructing a sequence of absolutely continuous controls which approximate a given continuous control. Essentially we take pathwise truncated left-mollifications.
We need to extend the class of potential maximizing control processes even more, because the examples of path-independent options show maximizing controls with a jump at the final time T. The example of the Up-and-Out Put teaches us, that we must admit a combination of singularly continuous processes *and* a final time jump. Moreover, the example of the realistic Up-and-Out Call indicates that jumps can happen basically any time. The result is

Theorem 3.3 *If F is lower semicontinuous, then the upper hedging price is*

$$\sup_{\lambda} I\!\!E\left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right],$$

where we maximize over all adapted, nondecreasing right-continuous control processes λ starting at zero.

In principle, the proof will be the same as in the previous extension to continuous controls, namely we approximate a given right-continuous control with jumps by continuous controls. The construction, however, is much more difficult, because we need to capture the jumps in the approximation without being allowed to look ahead (adaptivity!). The property that saves us is the right-continuity of the (augmented) Brownian filtration , which allows us to look infinitesimally far ahead and thus actually to predict a future jump with increasing precision as we get closer. The details of this construction are rather technical.

To formulate the last theorem, we need some extension of g, which is first of all *defined* even for path with jumps, secondly lower semicontinuous and thirdly agrees with the original g on continuous paths. Ideally we would like to have identifiable such g's. We give the details in the next section and advise the reader at this stage to ignore that g is potentially undefined.
We have extended the class of potential maximizers sufficiently much. The remaining problem we are worried about is, that a good assumption for existence of maximizers would be *upper semicontinuity* of the contingent claim g rather than *lower semicontinuity.* For continuous payoffs we don't have this problem and we get

Corollary 3.4 *For a large class of options g, including all path-independent options, options which depend* continuously *on the final stock-price, the maximal stock-price, the minimal stock-price and the average stock-price, the upper hedging price is*

$$\sup_{\lambda} I\!\!E\left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right],$$

where we maximize over all adapted, nondecreasing right-continuous control processes λ starting at zero.

Here, the payoff function g is defined even for stock price paths with jumps, in fact it is the same as the given g.
The assumption of continuity helps a lot to get nice results, but it is unfortunately not a realistic assumption. Digital and Barrier options for example don't fall under the class of options of our corollary. In the following part of the dissertation we examine how important lower semicontinuity actually is, and one result here is

Theorem 3.5 *If F is lower semicontinuous, then the upper hedging price depends lower semicontinuously on the initial stock price S_0 (no matter which of the above theorems we use to compute it).*

Our conjecture is that if we start with the upper semicontinuous version of F, the upper hedging price (as a function of the initial stock price S_0) is the upper semicontinuous version of the upper hedging price we get for the lower semicontinuous version of F. A clarificatin of this statement will be provided for path-independent options. We could cover path-dependent options only in the example section. Let $F = \phi(S_T)$. Then we prove

Theorem 3.6 *If ϕ is lower semicontinuous, then the face-lifted $\hat{\phi}$ is also lower semicontinuous.*

If ϕ is not lower semicontinuous, then face-lifting does not necessarily produce the correct value function (see our example of the Cactus Option). On the other hand, we would really like to take an upper semicontinuous ϕ, because then we get the following

Theorem 3.7 *If ϕ is upper semicontinuous, then there exists for each $x \in [0, \infty)$ a maximizing number $\lambda^*(x) \in [0, \infty]$ such that*

$$\hat{\phi}(x) = e^{-\alpha\lambda^*(x)}\phi(xe^{-\lambda^*(x)}).$$

Moreover, the maximizing control λ^* in the upper hedging price formula is given by

$$\lambda^*(t) = \mathbb{I}_{\{t=T\}}\lambda^*(S(T)).$$

The next theorem dissolves the dilemma, that on the one hand for existence of the maximizing control we need upper semicontinuity, and on the other hand, to get the upper hedging price right, it would be nice to have lower semicontinuity of the payoff. Certainly, we are in good shape if the payoff is continuous, but we can say more:

Theorem 3.8 *Let the lower semicontinuous version of a given payoff ϕ be*

$$\phi_*(x) \triangleq \inf_{x_n \to x} \liminf_{n\to\infty} \phi(x_n).$$

If $\hat{\phi} = \hat{\phi}_*$, then even starting with the not necessarily lower semicontinuous ϕ face-lifting produces the correct upper hedging price.
If, additionally, ϕ is upper semicontinuous, then the control problem admits a maximizer.

Its relevance is of practical nature: The two digital put options

(1) $\phi(x) = \mathbb{I}_{\{x<B\}}$ (this is lower semicontinuous.)

(2) $\phi(x) = \mathbb{I}_{\{x\leq B\}}$ (this is upper semicontinuous.)

should have the same $\hat{\phi}$, and in fact they do! The first one is the lower semicontinuous version of the second one. However, only the upper semicontinuous version admits a maximizing control.
We finally conject the existence of a maximizing control. To approach this problem we have found out that it would be most promising to clarify the one-to-one correspondence between the maximizing control and the cheapest hedge, because the latter has been proved to exist: It is essentially a stochastic integral representation result. We outline this one-to-one correspondence in the path-independent case to underline our conjecture.

4 The Theoretical Results in Detail

4.1 The Setup

We work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathbf{C}[0,T]$ is the space of continuous functions on the interval $[0,T]$ and $\mathbb{P}$ is the Wiener measure. A filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ can be created by setting $\mathcal{F}_t^0 \triangleq \sigma\{\omega : 0 \leq s \leq t\}$, which is the smallest σ-field which makes all projections measurable. $\mathcal{F}_t^0$ coincides with the Borel σ-field generated by the sup-norm topology (Parthasarathy Thm 2.1). We let $\mathcal{F}_t$ be the augmentation of $\mathcal{F}_t^0$ and $\mathcal{F} \triangleq \mathcal{F}_T$. We define the following spaces of controls

$$\begin{aligned}
\Lambda_+^{\mathrm{rc}} &\triangleq \{\lambda : [0,T] \to [0,\infty) | \lambda(0) = 0, \lambda \text{ is nondecreasing} \\
&\quad \text{and right-continuous}\} && (3) \\
\Lambda_+^{\mathrm{c}} &\triangleq \{\lambda : [0,T] \to [0,\infty) | \lambda(0) = 0, \lambda \text{ nondecreasing, continuous}\} && (4) \\
\Lambda_+^{\mathrm{ac}} &\triangleq \{\lambda \in \Lambda_+^{\mathrm{c}} | \lambda \text{ is absolutely continuous }\} && (5) \\
\Lambda_+^{\mathrm{acb}} &\triangleq \{\lambda \in \Lambda_+^{\mathrm{ac}} | \lambda' \text{ is bounded}\} && (6)
\end{aligned}$$

The subscript + stands for "nondecreasing". This notation is chosen, because we can consider the space Λ_+ as a space of finite measures on the interval $[0,T]$ corresponding to shortselling constraints. Other spaces, say Λ_-, whose elements can be viewed as the negatives of measures, correspond to borrowing constraints, and a general spaces Λ would work for convex constraints. In this thesis, however, we restrict our attention to shortselling constraints.
These spaces are obviously related in the following way:

$$\Lambda_+^{\mathrm{rc}} \supset \Lambda_+^{\mathrm{c}} \supset \Lambda_+^{\mathrm{ac}} \supset \Lambda_+^{\mathrm{acb}} \tag{7}$$
$$\Omega \supset \Lambda_+^{\mathrm{c}} \supset \Lambda_+^{\mathrm{ac}} \supset \Lambda_+^{\mathrm{acb}} \tag{8}$$

Similarly we define the following spaces of control processes:

$$\begin{aligned}
\mathcal{L}_+^{\mathrm{rc}} &\triangleq \{\lambda : \Omega \times [0,T] \to [0,\infty] | \ \lambda \text{ is adapted,} && (9) \\
&\quad \lambda(\omega,T) < \infty \text{ and } \lambda(\omega,\cdot) \in \Lambda_+^{\mathrm{rc}} \text{ for } I\!\!P\text{-a.e. } \omega\} \\
\mathcal{L}_+^{\mathrm{rcb}} &\triangleq \{\lambda \in \mathcal{L}_+^{\mathrm{rc}} | \exists C > 0 \text{ such that } \lambda(\omega,T) \le C \text{ for } I\!\!P\text{-a.e. } \omega\} && (10) \\
\mathcal{L}_+^{\mathrm{c}} &\triangleq \{\lambda \in \mathcal{L}_+^{\mathrm{rc}} | \lambda(\omega,\cdot) \in \Lambda_+^{\mathrm{c}} \text{ for } I\!\!P\text{-a.e. } \omega\} && (11) \\
\mathcal{L}_+^{\mathrm{ac}} &\triangleq \{\lambda \in \mathcal{L}_+^{\mathrm{c}} | \lambda(\omega,\cdot) \in \Lambda_+^{\mathrm{ac}} \text{ for } I\!\!P\text{-a.e. } \omega\} && (12) \\
\mathcal{L}_+^{\mathrm{acb}} &\triangleq \{\lambda \in \mathcal{L}_+^{\mathrm{ac}} | \exists C > 0 : \lambda'(\omega,t) \in [0,C] \forall t \text{ for } I\!\!P\text{-a.e. } \omega\} && (13)
\end{aligned}$$

These spaces are obviously related in the following way:

$$\mathcal{L}_+^{\mathrm{rc}} \supset \mathcal{L}_+^{\mathrm{c}} \supset \mathcal{L}_+^{\mathrm{ac}} \supset \mathcal{L}_+^{\mathrm{acb}} \tag{14}$$

The adaptivity of a control process λ is an important condition, because we certainly don't want the control to know any future stock price. This adaptivity can be characterized in one of the following three equivalent ways:

(1) $\omega_1(s) = \omega_2(s)\ \forall s \le t \Rightarrow \lambda(\omega_1)(t) = \lambda(\omega_2)(t)$.

(2) For each $t \in [0,T]$, $\lambda(\omega)(t)$ is a function of $\{\omega(s) : 0 \le s \le t\}$.

(3) $B \in \mathcal{G}_t \Rightarrow \lambda^{-1}(B) \in \mathcal{F}_t$, where $\mathcal{G}_t$ is the σ-algebra generated by all paths $\lambda \in \Lambda_+$ upto time t.

Notice that $\lambda = 0$ is certainly in $\mathcal{L}_+$. We equip Λ_+^{rc} with the *topology of weak convergence*, which means that a sequence of functions $\{\lambda_n\}$ is said to converge to a function λ, if $\lim_n \lambda_n(t) = \lambda(t)$ for each continuity point t of λ *and* for $t \in \{0,T\}$. We refer to the appendix for a list of properties of weak convergence.

4.2 Contingent Claims

On the event space we are given a payoff

$$F : \mathbf{C}[0,T] \to [0,\infty) \tag{15}$$

We only assume $\mathcal{F}_T$-measurability and nonnegativity of F throughout. We call F *upper semicontinuous* if

$$\limsup_{n\to\infty} F(\omega_n) \leq F(\omega), \text{ whenever } \omega_{\rm n} \to \omega \text{ uniformly} \tag{16}$$

and similarly *lower semicontinuous* if

$$\liminf_{n\to\infty} F(\omega_n) \geq F(\omega), \text{ whenever } \omega_{\rm n} \to \omega \text{ uniformly.} \tag{17}$$

We assume a geometric Brownian motion as a model for the underlying instrument, called the“stock”:

$$S(t,\omega) \triangleq S_0 e^{\sigma\omega(t)+\mu t} \quad ; \mu \triangleq r - \frac{1}{2}\sigma^2 \tag{18}$$

r is the risk-free rate, $\sigma > 0$ the volatility of the stock. F will then be of the form

$$F(\omega) = g(S(\cdot,\omega)) \tag{19}$$

Upper or lower semicontinuity is not a natural property of contingent claims. It will turn out that we can only expect maximizing controls to exist if the contingent claim F is upper semicontinuous. On the other hand, most of our theory will only work for lower semicontinuous F. From a practical point of view this usually does not matter: For instance, an up-and-out call option normally knocks out when the barrier is *reached.* The upper semicontinuous version is the corresponding option that knocks out when the barrier is *crossed.* In a continuous model these subtle differences won't effect valuation and hedging.
Note that we started with a stock, whose mean rate of return is equal to the risk-free rate r. This is actually only true in the risk-neutral measure world. But in order to make this thesis easier to read, we have decided to place ourselves in this world upfront. We certainly could have started with a different mean rate of return, say b. But after changing the measure we would be back at $b = r$. This procedure has been discussed so many times, and this thesis has different priority. We refer the reader to the literature, e.g. [KARATZAS and SHREVE: Methods of Mathematical Finance]. In order to invoke convergence theorems we may sometimes want to assume the *exponential growth condition*:

$$0 \leq F(\omega) \leq C_1 + C_2 e^{\gamma \max_{0\leq t\leq T}\omega(t)} \tag{20}$$

for some nonnegative constants C_1, C_2 and γ. We will explicitely state, when it will be used. We may also assume that F is greater than zero on a set of positive probability. These assumptions are valid for a large class of options including vanilla puts, digital puts, Asian puts, lookback puts, barrier options, power options.
As we have seen in the introductory example of the up-and-out call option, the

maximizing controls may have jumps. Therefore we want to define a corresponding payoff on paths that have jumps. To do that we define the two functions $F^*, F_* : \mathbf{C}[0,T] \times \Lambda_+^{\text{rc}} \to [0,\infty)$:

$$F^*(\omega,\lambda) \triangleq \sup\left\{ \limsup_{n\to\infty} F(\omega - \frac{1}{\sigma}\lambda_n) \middle| \lambda_n \to \lambda \text{ weakly}, \lambda_n \in \Lambda_+^{\text{c}} \right\} \quad (21)$$

$$F_*(\omega,\lambda) \triangleq \inf\left\{ \liminf_{n\to\infty} F(\omega - \frac{1}{\sigma}\lambda_n) \middle| \lambda_n \to \lambda \text{ weakly}, \lambda_n \in \Lambda_+^{\text{c}} \right\} \quad (22)$$

4.3 Properties of the upper- and lower semicontinuous Versions of Contingent Claims

Theorem 4.1 *(F^*/F_*-**Extension**)*

(1) F^* is upper semicontinuous.

(2) F_* is lower semicontinuous.

(3) $F^*(\omega,\lambda) = F(\omega - \frac{1}{\sigma}\lambda)$ for $\lambda \in \Lambda_+^c$ and for upper semicontinuous F.

(4) $F_*(\omega,\lambda) = F(\omega - \frac{1}{\sigma}\lambda)$ for $\lambda \in \Lambda_+^c$ and for lower semicontinuous F.

(5) $F^* \geq F_*$.

Proof. We only do (1) and (3). (2) and (4) is completely analogous to (1) and (3) respectively and (5) is obvious. (1) is true in a more general scenario: Let (C,d) be a metric space, (C_0,d_0) a dense metric subspace with respect to the d-metric, not with respect to the d_0-metric. Let $F : (C_0,d_0) \to \mathbb{R}$ be upper semicontinuous, i.e. $\limsup F(q_n) \leq F(q)$, whenever $d_0(q_n,q) \to 0$. Define

$$F^* : (C,d) \to [0,\infty)$$

$$F^*(x) \triangleq \sup\left\{ \limsup_{n\to\infty} F(q_n) \middle| q_n \xrightarrow{d} x, q_n \in C_0 \right\}$$

One can imagine the scenario $C = \mathbb{R}$, $C_0 = \mathbb{Q}$ and $d = d_0$ can be the Euclidian distance. Our case is

$$\begin{aligned} C &= \mathbf{C}[0,T] \times \Lambda_+^{\text{rc}} \\ C_0 &= \mathbf{C}[0,T] \times \Lambda_+^{\text{c}} \\ d &= \sup \times \text{weak} \\ d_0 &= \sup \times \sup \end{aligned}$$

We will now show that F^* is upper semicontinuous (even if F is not): Let $x \in C$ be given, let $\{x_n\}_n$ be a sequence in C for which $x_n \xrightarrow{d} x$. We have to show that $\limsup F^*(x_n) \leq F^*(x)$. The idea of the proof is: For given $\epsilon > 0$ we construct a sequence $\{p_n\}_n$ in C_0 such that $p_n \xrightarrow{d} x$ and

$$|\limsup F^*(x_n) - \limsup F(p_n)| < \epsilon.$$

If then $\limsup F^*(x_n) > F^*(x)$, then let $\epsilon \triangleq \frac{1}{2}[F^*(x) - \limsup F^*(x_n)] > 0$, construct the sequence $\{p_n\}_n$ in C_0 for this ϵ and observe that then $\limsup F^*(p_n) > F^*(x)$ as well, which contradicts the definition of F^* being a supremum. Consequently, we must have $\limsup F^*(x_n) \leq F^*(x)$.

A. Choose a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that

$$\lim F^*(x'_n) = \limsup F^*(x_n).$$

B. For each n, choose a sequence $\{q_{k,n}\}_k$ in C_0 such that $q_{k,n} \xrightarrow{d} x'_n$ as $k \to \infty$ and

$$0 \leq F^*(x'_n) - \limsup_{k\to\infty} F(q_{k,n}) < \frac{1}{n}.$$

C. For each sequence $\{q_{k,n}\}_k$ choose a subsequence $\{q'_{k,n}\}_k$ such that

$$\lim_{k\to\infty} F(q'_{k,n}) = \limsup_{k\to\infty} F(q_{k,n}).$$

D. For each n, set $p_n \triangleq q'_{k_n,n}$, where k_n is chosen to guarantee

$$d(q'_{k_n,n}, x'_n) < \frac{1}{n} \quad \text{and}$$

$$|F(q'_{k_n,n}) - F^*(x'_n)| < \frac{2}{n}.$$

Now, on the one hand, this will imply that

$$d(p_n, x) \leq d(p_n, x'_n) + d(x'_n, x) < \frac{1}{n} + d(x'_n, x) \to 0 \quad \text{as } n \to \infty.$$

On the other hand,

$$|\limsup_{m\to\infty} F^*(x_m) - \limsup_{m\to\infty} F(p_m)| \leq$$

$$|\lim_{m\to\infty} F^*(x'_m) - F^*(x'_n)| + |F^*(x'_n) - F(q'_{k_n,n})| + |F(q'_{k_n,n}) - \lim_{m\to\infty} F(q'_{k_m,m})|.$$

This is less than any given $\epsilon > 0$ for a sufficiently large n. This completes the proof of (1). To prove (3) we assume further that $d(q_n, q) \to 0$ implies $d_0(q_n, q) \to 0$ for $q_n, q \in C_0$. This is true in our case due to theorem 6.9. Now, because F is upper semicontinuous, $F(q) \geq \limsup F(q_n)$, whenever $q_n \xrightarrow{d} q$. Therefore $F(q) \geq F^*(q)$. To get the other inequality, notice that q can be approximated by the constant sequence consisting only of q's, and then by the definition of the supremum, $F^*(q) \geq \limsup F(q) = \lim F(q) = F(q)$. This completes the proof of the theorem.

The exponential groth property for F^*/F_*

Since

$$\max_{t\in[0,T]} (\omega(t) - \frac{1}{\sigma}\lambda(t)) \leq \max_{t\in[0,T]} \omega(t) \quad \forall \lambda \in \Lambda_+^{\mathrm{rc}}$$

we conclude that

$$F^*(\omega, \lambda) \leq C_1 + C_2 e^{\gamma \max_{t\in[0,T]} \omega(t)} \quad \forall \lambda \in \Lambda_+^{\mathrm{rc}}.$$

Using the known density for the maximum of a Brownian Motion in $[0, T]$ [KARATZAS and SHREVE, Brownian Motion and Stochastic Calculus, Section 2.8.], we can compute its moment generating function

$$\int_\Omega e^{\gamma \max_{t\in[0,T]} \omega(t)} \, d\mathbb{P}(\omega) = 2e^{\frac{1}{2}\gamma^2 T} \mathcal{N}(\gamma\sqrt{T}),$$

which shows that both F^* and F_* are integrable.

We will call F^* the *upper semicontinuous version* of F and F_* the *lower semicontinuous version* of F. The question arises, whether we can identify F^* and F_*. It turns out that in several interesting cases the formula $F^*(\omega, \lambda) = F(\omega - \frac{1}{\sigma}\lambda)$ holds even for $\lambda \in \Lambda_+^{\mathrm{rc}}$, although $F(\omega - \frac{1}{\sigma}\lambda)$ is actually only defined for $\lambda \in \Lambda_+^{\mathrm{c}}$. We collect details in the following:

Theorem 4.2 *(F^*/F_*-**Identification**)*

(1) Let $F(\omega) = g(S(T, \omega))$ for some function $g : [0, \infty) \to [0, \infty)$.
If g is upper semicontinuous, then

$$F^*(\omega, \lambda) = g(S(T, \omega)e^{-\lambda(T)}).$$

If g is lower semicontinuous, then

$$F_*(\omega, \lambda) = g(S(T, \omega)e^{-\lambda(T)}).$$

In particular, if g is continuous ,then

$$F^*(\omega, \lambda) = F_*(\omega, \lambda) = g(S(T, \omega)e^{-\lambda(T)}).$$

(2) Let $F(\omega) = g(S(T, \omega), \max_{t\in[0,T]} S(t, \omega))$ for some function

$$g : [0, \infty) \times [0, \infty) \to [0, \infty)$$

.

If g is upper semicontinuous, then

$$F^*(\omega, \lambda) \leq g\left(S(T, \omega)e^{-\lambda(T)}, \sup_{t\in[0,T]} (S(t, \omega)e^{-\lambda(t)}\right).$$

If g is lower semicontinuous, then

$$F_*(\omega, \lambda) \geq g\left(S(T, \omega)e^{-\lambda(T)}, \sup_{t\in[0,T]} (S(t, \omega)e^{-\lambda(t)}\right).$$

In particular, if g is continuous ,then

$$F^*(\omega, \lambda) = F_*(\omega, \lambda) = g\left(S(T, \omega)e^{-\lambda(T)}, \sup_{t\in[0,T]} (S(t, \omega)e^{-\lambda(t)}\right).$$

Analogous statements hold for functions $g : [0, \infty)^4 \to [0, \infty)$ of the form

$$F(\omega) = g\left(S(T, \omega), \max_{t\in[0,T]} S(t, \omega), \min_{t\in[0,T]} S(t, \omega), \int_0^T S(t, \omega)\, dt\right)$$

(3) Let $F(\omega) = g(S(t_1, \omega), \ldots, S(t_N, \omega))$ for finitely many checkpoints $0 \leq t_1 < t_2 < \cdots < t_N \leq T$, N an integer and $g : [0, \infty)^N \to [0, \infty)$. If g is

upper semicontinuous and nonincreasing in each variable (except possibly in $S(T)$), then

$$F^*(\omega,\lambda) = g\left(S(t_1,\omega)e^{-\lambda(t_1)},\dots,S(t_N,\omega)e^{-\lambda(t_N)}\right).$$

If g is lower semicontinuous in each variable, then

$$F_*(\omega,\lambda) \le g\left(S(t_1,\omega)e^{-\lambda(t_1)},\dots,S(t_N,\omega)e^{-\lambda(t_N)}\right).$$

Denoting

- $$M(T) \triangleq \max_{t\in[0,T]} S(t)$$
- $$m(T) \triangleq \min_{t\in[0,T]} S(t)$$
- $$A(T) \triangleq \frac{1}{T}\int_0^T S(t)\,dt$$

this theorem covers at least

(1) all path-independent options $g(S(T))$

(2) an up and out call option: $g(S(T),M(T)) = (S(T)-K)^+ I\!I_{\{M(T)\le B\}}$

(3) a book of barrier options,
e.g. $g(S(T),M(T)) = \sum_j c_j(S(T)-K_j)^+ I\!I_{\{M(T)\le B_j\}}$

(4) a lookback put option: $g(S(T),M(T)) = M(T)-S(T)$

(5) an Asian put option $g(S(T),A(T)) = (A(T)-S(T))^+$

(6) a discrete Asian put : $g(S(t_1),\dots,S(t_N)) = (K - average\{S(t_i)\}_{i=1}^N)^+$

(7) an up and out call option, where it is only checked finitely many times, whether the option has knocked out or not: $g(S(t_1),\dots,S(t_N)) = (S(t_N)-K)^+ I\!I_{\{S(t_1)\le B\}}\cdot\dots\cdot I\!I_{\{S(t_N)\le B\}}$ for $t_N = T$ (which is the realistic version of an up and out call.)

(8) a quotient option like: $g(S_t,S_T) = \frac{K I\!I_{\{S_T\le B\}}}{S_t^\beta}$ for some nonnegative constants K, B and β and $t\in[0,T]$.

Proof.

(1) We have to show two inequalities:

$F^* \le g$:
If $\lambda_n \in \Lambda_+^{\mathrm{c}}$ converges weakly to $\lambda \in \Lambda_+^{\mathrm{rc}}$, then we always have convergence at the final time T. Therefore

$$S(T,\omega)e^{-\lambda_n(T)} \longrightarrow S(T,\omega)e^{-\lambda(T)}.$$

The upper semicontinuity of g implies for all $\lambda_n \xrightarrow{w} \lambda$

$$\limsup_{n\to\infty} g\left(S(T,\omega)e^{-\lambda_n(T)}\right) \leq g\left(S(T,\omega)e^{-\lambda(T)}\right).$$

Now taking the supremum over all such $\{\lambda_n\}_n$ on the left hand side of the inequality yields

$$F^*(\omega,\lambda) \leq g\left(S(T,\omega)e^{-\lambda(T)}\right).$$

$F^* \geq g$:

On the other hand, F^*, being the supremum, certainly satisfies

$$F^*(\omega,\lambda) \geq \limsup_{n\to\infty} g\left(S(T,\omega)e^{-\lambda_n(T)}\right)$$

for *all sequences* $\{\lambda_n\}_n$ in Λ_+^{c} converging weakly to λ. According to theorem 6.10, there exists a *"good" sequence* $\{\lambda_n\}_n$ in Λ_+^{c} converging weakly to λ, which additionally satisfies $\lambda_n(T) = \lambda(T)$ for all n. Taking this sequence in the above inequality, we can skip the limes superior and arrive at the result

$$F^*(\omega,\lambda) \geq g\left(S(T,\omega)e^{-\lambda(T)}\right).$$

The proof for F_* is analogous.

(2) We have to show the inequality: $F^* \leq g$:

If $\lambda_n \in \Lambda_+^{\mathrm{c}}$ converges weakly to $\lambda \in \Lambda_+^{\mathrm{rc}}$, then we always have convergence at the final time T. Therefore

$$S_0 \exp\{\sigma\omega(T) + \mu T - \lambda_n(T)\} \longrightarrow S_0 \exp\{\sigma\omega(T) + \mu T - \lambda(T)\}.$$

By theorem 6.11, additionally

$$\max_{t\in[0,T]}[S_0 \exp\{\sigma\omega(t) + \mu t - \lambda_n(t)\}] \longrightarrow \sup_{t\in[0,T]}[S_0 \exp\{\sigma\omega(t) + \mu t - \lambda(t)\}].$$

The upper semicontinuity of g implies for all $\lambda_n \xrightarrow{w} \lambda$

$$\limsup_{n\to\infty} g\left(S_0 e^{\sigma\omega(T)+\mu T-\lambda_n(T)}, \max_{t\in[0,T]}[S_0 e^{\sigma\omega(t)+\mu t-\lambda_n(t)}]\right)$$
$$\leq g\left(S_0 e^{\sigma\omega(T)+\mu T-\lambda(T)}, \sup_{t\in[0,T]}[S_0 e^{\sigma\omega(t)+\mu t-\lambda(t)}]\right).$$

Now taking the supremum over all such $\{\lambda_n\}_n$ on the left hand side of the inequality yields

$$F^*(\omega,\lambda) \leq g\left(S_0 e^{\sigma\omega(T)+\mu T-\lambda(T)}, \sup_{t\in[0,T]}[S_0 e^{\sigma\omega(t)+\mu t-\lambda(t)}]\right).$$

For a lower semicontinuous F we can do an analogous argument to complete the proof.

(3) We are trying to imitate the proof of (1). However, the $F^* \leq g$-part does not work anymore, because we do not know, whether for $t_k \notin \{0, T\}$, $\lambda_n(t_k)$ will converge to $\lambda(t_k)$. All we know is that all the accumulation points of the sequence $\{\lambda_n(t_k)\}_n$ are contained in the interval $[\lambda(t_k-), \lambda(t_k)]$ (theorem 6.3). To get the argument to work, we have to impose the stated condition that g is nonincreasing in each variable. Fortunately this is the case we are interested in, because if g was increasing in, say S_t, then the *delta* of this option would be positive at t, and so naturally one would not want to impose a shortselling constraint there. The proof of the first inequality is still based on the upper semicontinuity of g, whereas the proof for the other inequality is essentially the same as in (1). Here is how it works:

$F^* \leq g$: We want to show

$$F^*(\omega, \lambda) \leq g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda(t_N)}\right).$$

If this was not true, then there must be a sequence $\{\lambda_n\}_n$ in Λ_+^c converging weakly to λ, such that

$$\limsup_{n\to\infty} g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda_n(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda_n(t_N)}\right)$$
$$> g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda(t_N)}\right).$$

There exists a subsequence $\{\lambda_{n_j}\}_j$ of $\{\lambda_n\}_n$ such that

$$\limsup_{n\to\infty} g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda_n(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda_n(t_N)}\right)$$
$$= \lim_{j\to\infty} g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda_{n_j}(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda_{n_j}(t_N)}\right)$$

and, of course any further subsequence must have the same limit. We choose a further subsequence $\{\lambda_{n_{j_l}}\}_l$ such that for all $k = 1, 2, \ldots, N$

$$\lim_{l\to\infty} \lambda_{n_{j_l}}(t_k) = \eta_k \in [\lambda(t_k-), \lambda(t_k)].$$

This can be done by successively choosing further subsequences. Since $\eta_k \leq \lambda(t_k)$ for all k, we can deduce

$$S_0 e^{\sigma\omega(t_k)+\mu t_N-\eta_k} \geq S_0 e^{\sigma\omega(t_k)+\mu t_N-\lambda(t_k)}.$$

Finally we can put all the pieces together:

$$\lim_{l\to\infty} g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda_{n_{j_l}}(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda_{n_{j_l}}(t_N)}\right)$$
$$= \limsup_{n\to\infty} g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda_n(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda_n(t_N)}\right)$$
$$> g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda(t_N)}\right)$$
$$\geq g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\eta_1}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\eta_N}\right),$$

where the last step used the assumption that g is nonincreasing in each variable. This conclusion, however, violates the upper semicontinuity of g.

$F^* \geq g$: To show the other inequality, we only need the existence of a *"good" sequence* which keeps all the t_k fixed. This was done in theorem 6.10.

The existence of this *"good" sequence* also implies

$$F_*(\omega, \lambda) \leq g\left(S_0 e^{\sigma\omega(t_1)+\mu t_1-\lambda(t_1)}, \ldots, S_0 e^{\sigma\omega(t_N)+\mu t_N-\lambda(t_N)}\right).$$

The proof of the theorem is complete.
Remark: We do not expect to find a "good" sequence which is fixed at a countable number of times. If for instance it was fixed at all the rationals in $[0, T]$, then by continuity all the sequence members would have to be identical and could thus only converge to a given λ if this was already continuous.

4.4 Value Functions

To continue with the general theory we define value functions

$$u^*(S_0; \lambda) \triangleq \int_\Omega e^{-rT-\alpha\lambda(T,\omega)} F^*(\omega, \lambda(\omega))\, dI\!P(\omega) \tag{23}$$

$$u_*(S_0; \lambda) \triangleq \int_\Omega e^{-rT-\alpha\lambda(T,\omega)} F_*(\omega, \lambda(\omega))\, dI\!P(\omega) \tag{24}$$

for any $\lambda \in \mathcal{L}_+^{\mathrm{rc}}$ and

$$u(S_0; \lambda) \triangleq \int_\Omega e^{-rT-\alpha\lambda(T,\omega)} F(\omega - \frac{1}{\sigma}\lambda(\omega))\, dI\!P(\omega) \tag{25}$$

for any $\lambda \in \mathcal{L}_+^{\mathrm{c}}$. We notice that $0 \in \mathcal{L}_+^{\mathrm{rc}}$ and that $F^*(\omega, 0) = F_*(\omega, 0) = F(\omega)$ and so

$$u^*(S_0, 0) = u_*(S_0, 0) = u(S_0, 0) = I\!E[e^{-rT} F] \tag{26}$$

is the *Time Zero Value of the Unconstrained Contingent Claim*. We want to clarify in this thesis what is the *Time Zero Value of the Constrained Contingent Claim F*, also called the *upper hedging price* of F subject to the shortselling constraint. To do this we define the following value functions:

$$u^*(S_0) \triangleq \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} u^*(S_0; \lambda) \tag{27}$$

$$u_*(S_0) \triangleq \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} u_*(S_0; \lambda) \tag{28}$$

$$u(S_0; \mathrm{c}) \triangleq \sup_{\lambda \in \mathcal{L}_+^{\mathrm{c}}} u(S_0; \lambda) \tag{29}$$

$$u(S_0; \mathrm{ac}) \triangleq \sup_{\lambda \in \mathcal{L}_+^{\mathrm{ac}}} u(S_0; \lambda) \tag{30}$$

$$u(S_0; \mathrm{acb}) \triangleq \sup_{\lambda \in \mathcal{L}_+^{\mathrm{acb}}} u(S_0; \lambda) \tag{31}$$

$$\tag{32}$$

Let us first state the obvious relations between all the above value functions:

Theorem 4.3 *(Obvious Relations)*

$$\begin{aligned} u(S_0;\text{acb}) &\le u(S_0;\text{ac}) \le u(S_0;\text{c}) \\ u_*(S_0) &\le u^*(S_0) \\ u(S_0;\text{c}) &\le u^*(S_0) \\ u(S_0;\text{c}) &\le u_*(S_0) \text{ if } F \text{ is lower semicontinuous.} \end{aligned}$$

4.5 A Visible Version of the Main Hedging Result

Theorem 4.4 *If the contingent claim F satisfies the exponential growth condition, then $u(S_0;\text{acb})$ is the upper hedging price, i.e. if we start with initial wealth $u(S_0;\text{acb})$, we can construct a hedge which allows us to payoff F and respects the constraint that the leverage is bounded below by $-\alpha$ during the lifetime of the option ($\alpha \ge 0$). $u(S_0;\text{acb})$ is the minimal amount which permits this.*

Proof. This main hedging result has been proved in [KARATZAS and SHREVE: Methods of Mathematical Finance 5.6.2] in the following form: The upper hedging price is given by

$$\sup_{\lambda\in\mathcal{L}_+^{\text{acb}}} \mathbb{E}_\lambda \left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W_\lambda+\mu-\lambda}\right)\right]. \tag{33}$$

The Brownian motion W_λ has been defined to be

$$W_\lambda(t) \triangleq W(t) + \frac{1}{\sigma}\lambda(t), \tag{34}$$

and the expectation is to be taken under the probability measure $\mathbb{P}_\lambda$ defined as

$$\mathbb{P}_\lambda[A] \triangleq \int_A Z_\lambda(T)\, d\mathbb{P} \quad \forall\ A \in \mathcal{F}_T, \tag{35}$$

where the density process $Z_\lambda(t)$ is

$$Z_\lambda(t) \triangleq \exp\left\{-\frac{1}{\sigma}\int_0^t \lambda'(s)\, dW(s) - \frac{1}{2\sigma^2}\int_0^t (\lambda'(s))^2\, ds\right\}. \tag{36}$$

We have used the shorthand notation

$$g\left(S_0 e^{\sigma W_\lambda+\mu-\lambda}\right) \triangleq g\left(\left\{S_0 e^{\sigma W_\lambda(t)+(r-\frac{1}{2}\sigma^2)t-\lambda(t)}\right\}_{t\in[0,T]}\right). \tag{37}$$

This illuminates the importance of the space $\mathcal{L}_+^{\text{acb}}$, because in order to be even able to *state* the main hedging result, we have to ensure that the probability measure $\mathbb{P}_\lambda$ is defined. For that, it suffices that the process $\lambda'(t)$ is uniformly bounded, because it will turn $Z_\lambda(t)$ into a martingale. The other problem is that this formulation of the main hedging result really stretches our imagination: We are maximizing over changes of measure, a rather non-transparent procedure. Our new new version of the main hedging result basically says that we do not need the subscript λ at $\mathbb{E}$ and W, i.e. we claim

$$\begin{aligned} &\sup_{\lambda\in\mathcal{L}_+^{\text{acb}}} \mathbb{E}_\lambda \left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W_\lambda+\mu-\lambda}\right)\right] \\ =\ &\sup_{\lambda\in\mathcal{L}_+^{\text{acb}}} \mathbb{E} \left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right]. \end{aligned} \tag{38}$$

To show this we will now prove that even before taking the supremum

$$\mathbb{E}_\lambda\left[e^{-\alpha\lambda(T)}g\left(S_0 e^{\sigma W_\lambda+\mu-\lambda}\right)\right] = \mathbb{E}\left[e^{-\alpha\lambda(T)}g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right]. \tag{39}$$

The key observation for the proof is that it does not matter on which probability space $(\Omega, \mathcal{F}, \mathbb{P}_\lambda, \{\mathcal{F}_t\}_{t\in[0,T]})$ W_λ is a Brownian motion. We can take *any* Brownian motion on *any* probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]})$, and to make this concrete we define

$$\begin{aligned}
\tilde{\Omega} &\triangleq \Omega & (40)\\
\tilde{\mathcal{F}}_t &\triangleq \mathcal{F}_t & (41)\\
\tilde{\lambda}(\omega, t) &\triangleq \lambda(\omega, t) & (42)\\
\tilde{W}(t) &\triangleq W(t) - \frac{1}{\sigma}\int_0^t \lambda'(s)\,ds = W(t) - \frac{1}{\sigma}\lambda(t) & (43)\\
\tilde{\mathbb{P}}[A] &\triangleq \int_A \exp\left\{\frac{1}{\sigma}\int_0^T \lambda'(s)\,dW(s) - \frac{1}{2\sigma^2}\int_0^T (\lambda'(s))^2\,ds\right\} d\mathbb{P} & (44)
\end{aligned}$$

for all $A \in \mathcal{F}_T$. By Girsanov's change of measure theorem, $\tilde{W}$ is a Brownian motion on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]})$, adapted to the filtration $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$. Moreover, $\tilde{\lambda} \in \mathcal{L}_+^{\mathrm{acb}}$. We take this Brownian motion and this probability space to start with and follow the definitions of [KARATZAS and SHREVE]:

$$\begin{aligned}
\tilde{W}_{\tilde{\lambda}}(t) &\triangleq \tilde{W}(t) + \frac{1}{\sigma}\tilde{\lambda}(t)\\
&= \tilde{W}(t) + \frac{1}{\sigma}\lambda(t)\\
&= W(t) & (45)
\end{aligned}$$

The probability measure $\tilde{\mathbb{P}}_{\tilde{\lambda}}$ happens to agree with $\mathbb{P}$, because for all $A \in \mathcal{F}_T$ we have

$$\begin{aligned}
\tilde{\mathbb{P}}_{\tilde{\lambda}}[A] &= \int_A \exp\left\{-\frac{1}{\sigma}\int_0^T \tilde{\lambda}'(s)\,d\tilde{W}(s) - \frac{1}{2\sigma^2}\int_0^T (\tilde{\lambda}'(s))^2\,ds\right\} d\tilde{\mathbb{P}}\\
&= \int_A \exp\left\{-\frac{1}{\sigma}\int_0^T \tilde{\lambda}'(s)\,d\tilde{W}(s) - \frac{1}{2\sigma^2}\int_0^T (\tilde{\lambda}'(s))^2\,ds\right.\\
&\qquad \left. + \frac{1}{\sigma}\int_0^T \tilde{\lambda}'(s)\,dW(s) - \frac{1}{2\sigma^2}\int_0^T (\tilde{\lambda}'(s))^2\,ds\right\} d\mathbb{P}\\
&= \int_A \exp\left\{-\frac{1}{\sigma}\int_0^T \lambda'(s)\,dW(s) + \frac{1}{\sigma^2}\int_0^T (\lambda'(s))^2\,ds\right.\\
&\qquad -\frac{1}{2\sigma^2}\int_0^T (\lambda'(s))^2\,ds\\
&\qquad \left. + \frac{1}{\sigma}\int_0^T \lambda'(s)\,dW(s) - \frac{1}{2\sigma^2}\int_0^T (\lambda'(s))^2\,ds\right\} d\mathbb{P}\\
&= \mathbb{P}[A].
\end{aligned}$$

This proves equation 39 and the proof of the theorem is complete. We can now view the constraint valuation problem as a maximization over processes with absolutely continuous nondecreasing paths that have bounded derivatives. The control problem has been visualized! The fact that *all* our examples show maximizing control processes that have *singular* nondecreasing paths motivates us to go on.

4.6 Extension to Continuous Controls

We will now examine under which conditions the inequalities of the obvious relations can be reversed and whether they can be strict. It has been indicated in [KARATZAS and SHREVE: Methods of Mathematical Finance], that $u(S_0; \mathrm{acb}) = u(S_0; \mathrm{ac})$. Here is our new first basic result in this direction:

Theorem 4.5 *If F is lower semicontinuous, then*

$$u(S_0; \mathrm{acb}) \geq u(S_0; \mathrm{c}).$$

If F is not lower semicontinuous, then it can happen, that

$$u(S_0; \mathrm{acb}) < u(S_0; \mathrm{c}).$$

Proof. We must show that for any $\lambda \in \mathcal{L}_+^{\mathrm{c}}$

$$\sup_{\lambda \in \mathcal{L}_+^{\mathrm{acb}}} I\!E\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right] \geq I\!E\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right]. \tag{46}$$

Let $\lambda \in \mathcal{L}_+^{\mathrm{c}}$ be given. Define a sequence $\{\lambda_n\}_n$ of truncated left-mollifications of λ by

$$\lambda_n(t,\omega) \triangleq \int_{-1}^{0} \min\left[n, \lambda(t + \frac{u}{n}, \omega)\right] \varphi(u)\, du, \tag{47}$$

where the weight function $\varphi : I\!R \to [0, \infty)$ is given by

$$\varphi(t) \triangleq C_\varphi \exp\left\{\frac{-1}{(2t+1)^2 - 1}\right\} I\!I_{[-1,0]}(t). \tag{48}$$

C_φ is chosen such that $\int \varphi(t)\, dt = 1$. φ is then a nonnegative probability density function in $\mathbf{C}^\infty(I\!R)$ with support $[-1, 0]$. We shall see below that $\lambda_n \in \mathcal{L}_+^{\mathrm{acb}}$ and that λ_n converges to λ uniformly for all ω. Using this we can first say that for all n

$$\sup_{\lambda \in \mathcal{L}_+^{\mathrm{acb}}} I\!E\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right] \overset{\lambda_n \in \mathcal{L}_+^{\mathrm{acb}}}{\geq} I\!E\left[e^{-\alpha\lambda_n(T)} F(W - \frac{1}{\sigma}\lambda_n)\right] \tag{49}$$

and therefore

$$\begin{aligned}
\sup_{\lambda \in \mathcal{L}_+^{\mathrm{acb}}} I\!E\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right] &\geq \liminf_{n\to\infty} I\!E\left[e^{-\alpha\lambda_n(T)} F(W - \frac{1}{\sigma}\lambda_n)\right] \\
&\overset{\text{Fatou}}{\geq} I\!E\left[\liminf_{n\to\infty} e^{-\alpha\lambda_n(T)} F(W - \frac{1}{\sigma}\lambda_n)\right] \\
&= I\!E\left[e^{-\alpha\lambda(T)} \liminf_{n\to\infty} F(W - \frac{1}{\sigma}\lambda_n)\right] \\
&\overset{\text{F lsc}}{\geq} I\!E\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right].
\end{aligned}$$

To complete the proof, we must verify the following checklist for λ_n:

A. $\lambda_n(0) = 0$.

B. λ_n is adapted.

C. $\lambda_n \leq \lambda$.

D. $\lambda_n < \infty$ a.s.

E. λ_n is nondecreasing in t.

F. λ_n is absolutely continuous.

G. λ_n' is bouded by $n^2\varphi(-\frac{1}{2})$ uniformly in (t, ω).

H. λ_n is nondecreasing in n for each (t, ω).

I. $\lim \lambda_n(t) = \lambda(t)$ for all t and for a.e. ω.

J. λ_n converges to λ uniformly in t for a.e. ω.

Let's go through this checklist again, now including the proofs:

A. $\lambda_n(0) = 0$ is certainly true, if we use the natural extension of λ: $\lambda(t) = 0$ for $t < 0$.

B. λ_n is adapted, because λ is adapted and $\lambda_n(t)$ requires only knowledge of λ up to time t.

C. Since $t + \frac{u}{n} \leq t$ and λ is nondecreasing, we get for all ω

$$\begin{aligned} \lambda_n(t) = \int_{-1}^{0} \min\left[n, \lambda(t + \frac{u}{n})\right] \varphi(u)\, du \quad &\leq \quad \int_{-1}^{0} \lambda(t + \frac{u}{n})\varphi(u)\, du \\ &\leq \quad \int_{-1}^{0} \lambda(t)\varphi(u)\, du = \lambda(t). \end{aligned}$$

D. $\lambda_n < \infty$ follows from C: $\lambda_n \leq \lambda < \infty$ a.s.

E. λ_n is nondecreasing in t: Let $0 \leq t \leq s \leq T$. Since λ is assumed to be nondecreasing, we obtain for all $u \in [-1, 0]$

$$\lambda(t + \frac{u}{n}) \leq \lambda(s + \frac{u}{n})$$

and hence for all $u \in [-1, 0]$

$$\min\left[n, \lambda(t + \frac{u}{n})\right] \leq \min\left[n, \lambda(s + \frac{u}{n})\right].$$

This implies for all ω

$$\int_{-1}^{0} \min\left[n, \lambda(t + \frac{u}{n})\right] \varphi(u)\, du \leq \int_{-1}^{0} \min\left[n, \lambda(s + \frac{u}{n})\right] \varphi(u)\, du$$

F. λ_n is absolutely continuous. After the change of variables $t + \frac{u}{n} = y$ we get

$$\begin{aligned}
\lambda_n(t,\omega) &= \int_{-1}^{0} \min\left[n, \lambda(t+\frac{u}{n},\omega)\right] \varphi(u)\,du \\
&= n\int_{t-\frac{1}{n}}^{t} \min\left[n, \lambda(y,\omega)\right] \varphi(n(y-t))\,dy \\
&= n\int_{-\infty}^{T} \min\left[n, \lambda(y,\omega)\right] \varphi(n(y-t))\,dy,
\end{aligned}$$

which shows that $\lambda_n(t,\omega) \in \mathbf{C}^\infty[0,T]$. In particular, it is absolutely continuous.

G. $\lambda_n' \geq 0$ follows from E. and F. To get the upper bound, first observe that for any nondecreasing $\tilde{\lambda}$

$$\begin{aligned}
&\int_{t-\frac{1}{n}}^{t} \tilde{\lambda}(y)\varphi'(n(y-t))\,dy \\
= &\int_{t-\frac{1}{n}}^{t-\frac{1}{2n}} \tilde{\lambda}(y)\overbrace{\varphi'(n(y-t))}^{\geq 0}\,dy + \int_{t-\frac{1}{2n}}^{t} \tilde{\lambda}(y)\overbrace{\varphi'(n(y-t))}^{\leq 0}\,dy \\
\geq &\int_{t-\frac{1}{n}}^{t-\frac{1}{2n}} \tilde{\lambda}(t-\frac{1}{n})\varphi'(n(y-t))\,dy + \int_{t-\frac{1}{2n}}^{t} \tilde{\lambda}(t)\varphi'(n(y-t))\,dy \\
= &\frac{1}{n}\tilde{\lambda}(t-\frac{1}{n})\left[\varphi(-\frac{1}{2}) - \overbrace{\varphi(-1)}^{=0}\right] + \frac{1}{n}\tilde{\lambda}(t)\left[\overbrace{\varphi(0)}^{=0} - \varphi(-\frac{1}{2})\right] \\
= &-\frac{1}{n}\varphi(-\frac{1}{2})\left[\tilde{\lambda}(t) - \tilde{\lambda}(t-\frac{1}{n})\right]
\end{aligned}$$

Using this inequality for the nondecreasing $\tilde{\lambda}(t) \triangleq \min[n, \lambda(t)]$ we conclude that for all (t,ω)

$$\begin{aligned}
\lambda_n'(t,\omega) &= -n^2\int_{t-\frac{1}{n}}^{t} \min\left[n, \lambda(y,\omega)\right] \varphi'(n(y-t))\,dy \\
&\leq n\varphi(-\frac{1}{2})\left[\min[n,\lambda(t,\omega)] - \min[n,\lambda(t-\frac{1}{n},\omega)]\right] \\
&\leq n^2\varphi(-\frac{1}{2}).
\end{aligned}$$

H. λ_n is nondecreasing in n for each (t,ω): Since $\frac{1}{1+n} < \frac{1}{n}$, we know that

$$t + \frac{u}{n+1} > t + \frac{u}{n} \quad \forall u \in [-1,0].$$

Since furthermore λ is assumed to be nondecreasing, this implies

$$\lambda\left(t + \frac{u}{n+1}\right) \geq \lambda\left(t + \frac{u}{n}\right) \quad \forall\, u \in [-1,0]$$

and thus

$$\min\left[n, \lambda\left(t + \frac{u}{n+1}\right)\right] \geq \min\left[n, \lambda\left(t + \frac{u}{n}\right)\right] \quad \forall\, u \in [-1, 0]$$

It follows that

$$\begin{aligned}
\lambda_{n+1}(t) &= \int_{-1}^{0} \min\left[n+1, \lambda(t + \frac{u}{n+1})\right] \varphi(u)\, du \\
&\geq \int_{-1}^{0} \min\left[n, \lambda(t + \frac{u}{n+1})\right] \varphi(u)\, du \\
&\geq \int_{-1}^{0} \min\left[n, \lambda(t + \frac{u}{n})\right] \varphi(u)\, du \\
&= \lambda_n(t).
\end{aligned}$$

I. $\lim \lambda_n(t) = \lambda(t)$ for all t and for a.e. ω: Since $\lambda(T) < \infty$ a.s., we see that a.s.

$$\lim_{n\to\infty} \min\left[n, \lambda(t + \frac{u}{n})\right] = \lim_{n\to\infty} \lambda(t + \frac{u}{n}).$$

Using this and the Monotone Convergence Theorem, we obtain for a.e. ω

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-1}^{0} \min\left[n+1, \lambda(t + \frac{u}{n+1})\right] \varphi(u)\, du \\
&= \int_{-1}^{0} \lim_{n\to\infty} \min\left[n, \lambda(t + \frac{u}{n+1})\right] \varphi(u)\, du \\
&= \int_{-1}^{0} \lim_{n\to\infty} \lambda(t + \frac{u}{n}) \varphi(u)\, du \\
&= \int_{-1}^{0} \lambda(t) \varphi(u)\, du \\
&= \lambda(t).
\end{aligned}$$

J. λ_n converges to λ uniformly in t for a.e. ω, because pointwise convergence on a compact interval to a continuous function is uniform if the convergence is monotone. (*Dini's theorem*)

The upper semicontinuous Up-and-Out Call

$$F = (S(T) - K)^+ \mathbb{I}_{\{S(t) \leq B \;\; \forall t \in [0,T]\}}$$

serves as an example for strict inequality: If the staring point $S(0)$ is *at* the barrier, then

$$0 = u(B; \text{acb}) < u(B; \text{c}) = v(0, B; \alpha).$$

A singularly continuous control can create reflection at the barrier and save the option from knocking out. No absolutely continuous control, however, can save the option from knocking out instantly. We also see here, that, had we taken the lower semicontinuous Up-and-Out Call, this difference would not appear. This completes the proof of the theorem.

4.7 Extension to Right-Continuous Controls

So far we have learned that we can maximize over (possibly singularly) continuous control processes to find the upper hedging price. All the examples of path-independent options, however, show maximizing controls with a jump at the final time T. The example of the Up-and-Out Put teaches us, that we must admit a combination of singularly continuous processes *and* a final time jump. Moreover, the example of the realistic Up-and-Out Call indicates that jumps can happen basically any time. This motivates the importance of the next theorem. In principle, the proof will be the same as in the previuos Extension to Continuous Controls, namely we approximate a given right-continuous control with jumps by continuous controls. The construction, however, is much more difficult, because we need to capture the jumps in the approximation without being allowed to lood ahead (adaptivity!). The property that saves us is the right-continuity of the (augmented) Brownian filtration [KARATZAS and SHREVE 2.7.7], which allows us to look infinitesimally far ahead and thus actually to predict a future jump with increasing precision as we get closer. The details of this construction are rather technical.

Theorem 4.6

$$u(S_0; \mathrm{c}) = u_*(S_0).$$

For the proof we need the following

Lemma 4.7 *For a given $\lambda \in \mathcal{L}_+^{rcb}$ there exists a positive constant C and a random function*

$$\psi : \Omega \times [0,T] \times [0,T] \longrightarrow [0,C]$$

such that for every $\omega \in \Omega$

(a) $s \mapsto \psi(s,t)$ is continuous for all $t \in [0,T]$,

(b) $t \mapsto \psi(s,t)$ is RCLL and nondecreasing for all $s \in [0,T]$

and for every $s,t \in [0,T]$ and a.e. ω

$$\psi(s,t) = I\!E[\lambda(t)|\mathcal{F}(s)].$$

Proof of theorem 4.6. Since $u(S_0; \mathrm{c}) \le u_*(S_0)$ is obvious, we must show here that for any $\lambda \in \mathcal{L}_+^{\mathrm{rc}}$

$$\sup_{\lambda \in \mathcal{L}_+^{\mathrm{c}}} I\!E\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right] \ge I\!E\left[e^{-\alpha\lambda(T)} F_*(W,\lambda)\right].$$

Let $\lambda \in \mathcal{L}_+^{\mathrm{rc}}$ be given. We can do the same argument as in the previous theorem 4.5, if we can create a sequence $\{\lambda_n\}_n$ in $\mathcal{L}_+^{\mathrm{rc}}$ satisfying the the following condition: λ_n is adapted and there exists a *singular exceptional set* $N \subset \Omega$, $I\!P[N] = 0$, and for all $\omega \in \Omega \setminus N$:

A. $\lambda_n(0) = 0$.

B. λ_n is nondecreasing in t.

C. λ_n is continuous.

D. λ_n converges weakly to λ.

With such a sequence we can first say that for all n

$$\sup_{\lambda \in \mathcal{L}_+^c} \mathbb{E}\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right] \overset{\lambda_n \in \mathcal{L}_+^c}{\geq} \mathbb{E}\left[e^{-\alpha\lambda_n(T)} F(W - \frac{1}{\sigma}\lambda_n)\right] \tag{50}$$

and therefore

$$\begin{aligned}
\sup_{\lambda \in \mathcal{L}_+^c} \mathbb{E}\left[e^{-\alpha\lambda(T)} F(W - \frac{1}{\sigma}\lambda)\right] &\geq \liminf_{n\to\infty} \mathbb{E}\left[e^{-\alpha\lambda_n(T)} F(W - \frac{1}{\sigma}\lambda_n)\right] \\
&\overset{\text{Fatou}}{\geq} \mathbb{E}\left[\liminf_{n\to\infty} e^{-\alpha\lambda_n(T)} F(W - \frac{1}{\sigma}\lambda_n)\right] \\
&= \mathbb{E}\left[e^{-\alpha\lambda(T)} \liminf_{n\to\infty} F(W - \frac{1}{\sigma}\lambda_n)\right] \\
&\overset{\text{Def of } F_*}{\geq} \mathbb{E}\left[e^{-\alpha\lambda(T)} F_*(W, \lambda)\right]
\end{aligned}$$

To complete the proof, we create a suitable sequence λ_n: We assume first that $\lambda(T) \leq C$ a.s. for some positive constant $C \in \mathbb{R}$. Now we take the random variable ψ of the lemma and define ω-wise

$$\gamma_n(t) \triangleq \psi\left((t - \frac{1}{n})^+, t\right), \tag{51}$$

$$\check{\lambda}_n(t) \triangleq \int_0^1 \gamma_n(t + \frac{u}{n})\varphi(u)\,du - \int_0^1 \gamma_n(\frac{u}{n})\varphi(u)\,du \tag{52}$$

$$\lambda_n(t) \triangleq \max_{0\leq s\leq t} \check{\lambda}_n(s), \tag{53}$$

where φ is a probability density function in $\mathbf{C}^\infty(\mathbb{R})$ with support $[0,1]$. With this definition λ_n is adapted. Now we have to go through the checklist:

A. $\lambda_n(0) = 0$ is trivial.

B. λ_n has been forced to be nondecreasing in t.

C. $\lambda_n(t)$ is continuous in t: We only need to check that

$$\begin{aligned}
\check{\lambda}_n(t) &= \int_0^1 \gamma_n(t + \frac{u}{n})\varphi(u)\,du \\
&= n\int_{-\infty}^{\infty} \gamma_n(y)\varphi(n(y-t))\,dy
\end{aligned}$$

is continuous in t. This follows from the continuity of φ and the Bounded Convergence Theorem.

D. λ_n converges weakly to λ: First of all,

$$\int_0^1 \gamma_n(\frac{u}{n})\varphi(u)\,du = \int_0^1 \psi(0, \frac{u}{n})\varphi(u)\,du \tag{54}$$

is a constant function converging to zero. Then

$$\begin{aligned}
\liminf_{n\to\infty} \check{\lambda}_n(t) &\overset{\text{Fatou}}{\geq} \int_0^1 \liminf_{n\to\infty} \gamma_n(t+\frac{u}{n})\varphi(u)\,du \\
&\overset{\text{Def. of } \gamma_n}{=} \int_0^1 \liminf_{n\to\infty} \psi\left((t+\frac{u-1}{n})^+, t+\frac{u}{n}\right)\varphi(u)\,du \\
&\overset{t\mapsto\psi(s,t) \text{ is } \uparrow}{\geq} \int_0^1 \liminf_{n\to\infty} \psi\left((t+\frac{u-1}{n})^+, t\right)\varphi(u)\,du \\
&\overset{s\mapsto\psi(s,t) \text{ is cont.}}{=} \int_0^1 \psi\,(t,t)\,\varphi(u)\,du \\
&= \psi\,(t,t) \qquad (55)
\end{aligned}$$

On the other hand, for fixed $k \in \mathbb{N}$ and all $n \geq k$ we get

$$\psi\left((t+\frac{u-1}{n})^+, t+\frac{u}{n}\right) \leq \psi\left((t+\frac{u-1}{n})^+, t+\frac{u}{k}\right),$$

because the mapping $t \mapsto \psi(s,t)$ is nondecreasing. We take the limes superior on both sides and get

$$\begin{aligned}
\limsup_{n\to\infty} \gamma_n(t+\frac{u}{n}) &\leq \limsup_{n\to\infty} \psi\left((t+\frac{u-1}{n})^+, t+\frac{u}{k}\right) \\
&\leq \psi\left(t, t+\frac{u}{k}\right)
\end{aligned}$$

Recalling that ψ is bounded above by C this yields for all k

$$\begin{aligned}
\limsup_{n\to\infty} \check{\lambda}_n(t) &\overset{\text{Fatou}}{\leq} \int_0^1 \limsup_{n\to\infty} \gamma_n(t+\frac{u}{n})\varphi(u)\,du \\
&\overset{t\mapsto\psi(s,t) \text{ is cont.}}{\leq} \int_0^1 \psi\left(t, t+\frac{u}{k}\right)\varphi(u)\,du.
\end{aligned}$$

We may now take the limes superior on the right hand side, use the right-continuity of $t \mapsto \psi(s,t)$ and get

$$\begin{aligned}
\limsup_{n\to\infty} \check{\lambda}_n(t) &\leq \limsup_{k\to\infty} \int_0^1 \psi\left(t, t+\frac{u}{k}\right)\varphi(u)\,du \\
&\overset{\text{Fatou}}{\leq} \int_0^1 \limsup_{k\to\infty} \psi\left(t, t+\frac{u}{k}\right)\varphi(u)\,du \\
&= \psi\,(t,t) \qquad (56)
\end{aligned}$$

As a combined result, we have derived

$$\begin{aligned}
\lim_{n\to\infty} \check{\lambda}_n(t) &= \psi(t,t) \qquad (57) \\
&\overset{\text{a.s.}}{=} \mathbb{E}[\lambda(t)|\mathcal{F}(t)] \\
&\overset{\text{a.s.}}{=} \lambda(t). \qquad (58)
\end{aligned}$$

It remains to show that $\lambda_n(t)$ converges to $\lambda(t)$ for all $t \in [0,T]$ a.s. as well. Since $\lambda_n(t) \geq \check{\lambda}_n(t)$, we get

$$\liminf_{n\to\infty} \lambda_n(t) \geq \liminf_{n\to\infty} \check{\lambda}_n(t) = \lambda(t) \qquad (59)$$

for free. What we need to complete D is

$$\limsup_{n\to\infty} \lambda_n(t) \le \lambda(t). \tag{60}$$

Let us fix t and choose for each n a number $s_n \in [0,t]$ such that

$$\check{\lambda}_n(s_n) = \lambda_n(t). \tag{61}$$

We also choose a subsequence $\{\lambda_{n_k}\}_k$ of $\{\lambda_n\}_n$ such that

$$\limsup_{n\to\infty} \lambda_n(t) = \lim_{k\to\infty} \lambda_{n_k}(t). \tag{62}$$

Consider the sequence $\{s_{n_k}\}_k$. It is bounded and therefore contains a subsequence $\{s_{n_{k_j}}\}_j$ converging to a number $s^* \in [0,t]$. Now we can put the pieces together and conclude

$$\begin{aligned} \limsup_{n\to\infty} \lambda_n(t) &= \lim_{k\to\infty} \lambda_{n_k}(t) \\ &= \lim_{j\to\infty} \lambda_{n_{k_j}}(t) \\ &= \lim_{j\to\infty} \check{\lambda}_{n_{k_j}}(s_{n_{k_j}}) \\ &\le \lambda(s^*) \\ &\le \lambda(t) \end{aligned}$$

The second last inequality follows from theorem 6.4 and the previous result that $\check{\lambda}_n$ converges pointwise to λ, and the last one from the fact that λ is nondecreasing.

The verification of the checklist is complete and we have derived the intermediate result

$$u(S_0;\mathrm{c}) = \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rcb}}} I\!\!E\left[e^{-\alpha\lambda(T)}F_*(W,\lambda)\right]. \tag{63}$$

Finally we must examine what happens if $\lambda(T) \le C$ a.s. does not hold, but rather $\lambda(T) < \infty$ a.s. In this case we take the given λ and look at the truncated approximation $\lambda_n \triangleq n \wedge \lambda$. Then we see that $\lambda_n \in \mathcal{L}_+^{\mathrm{rcb}}$ and λ_n converges weakly to λ. Since F_* is lower semicontinuous, our standard proof will take care of the rest. The proof of the theorem is complete.

Proof of Lemma 4.7. Let $\lambda \in \mathcal{L}_+^{\mathrm{rcb}}$ be given. Then first of all there exists a positive constant C such that

$$\lambda(T) \le C \text{ a.s.} \tag{64}$$

For $\lambda \in \Lambda_+^{\mathrm{rc}}$ and $\delta > 0$ define the *right modulus of continuity*

$$m_\delta(\lambda) \triangleq \sup_{0\le t\le T-\delta} [\lambda(t+\delta) - \lambda(t)]. \tag{65}$$

[BILLINGSLEY, Convergence of Probability Measures] shows that

$$\lim_{\delta\downarrow 0} m_\delta(\lambda) = 0 \;\; \forall \;\; \lambda \in \Lambda_+^{\mathrm{rc}}. \tag{66}$$

Equation 64 implies that the random variable $m_\delta(\lambda)$ satisfies

$$m_\delta(\lambda) \leq C \text{ a.s.} \quad \forall \ \ \delta > 0. \tag{67}$$

The bounded, nonnegative martingale $\{\mathbb{E}[m_\delta(\lambda)|\mathcal{F}(s)]\}_{0\leq s\leq T}$ has a RCLL modification which we call $M_\delta(s)$, i.e., for each $s \in [0,T]$

$$M_\delta(s) = \mathbb{E}[m_\delta(\lambda)|\mathcal{F}(s)] \text{ a.s.} \tag{68}$$

and without loss of generality, we take *every* path of M_δ to be RCLL, nonnegative and bounded above by C. We set

$$M^*_\delta \stackrel{\Delta}{=} \sup_{0\leq s\leq T} M_\delta(s) \leq C. \tag{69}$$

and without loss of generality assume

$$M^*_{\delta_1}(\omega) \leq M^*_{\delta_2}(\omega) \leq C \ \forall \ \omega \in \Omega, \forall \ \delta_1, \delta_2 \in (0,\infty) \cap \mathbb{Q}, \ \ \delta_1 < \delta_2. \tag{70}$$

(The set where equation 70 fails has probability zero; redefine $M_\delta(\cdot) \equiv 0$ on this set.) Note from *Doob's maximal martingale inequality* that

$$\mathbb{E}\left[(M^*_\delta)^2\right] \leq 4\mathbb{E}\left[(m_\delta(\lambda))^2\right] \stackrel{\delta\downarrow 0}{\longrightarrow} 0 \tag{71}$$

because of equations 66 and 67. We may thus choose a sequence of positive numbers $\{\delta_k\}_k$ with $\delta_k \downarrow 0$ such that

$$\sum_{k=1}^{\infty} k^2 \mathbb{E}\left[(m_{\delta_k}(\lambda))^2\right] < \infty. \tag{72}$$

Chebyshev's inequality implies

$$\mathbb{P}\left[M^*_{\delta_k} \geq \frac{1}{k}\right] \leq k^2 \mathbb{E}\left[(M^*_{\delta_k})^2\right] \leq 4k^2 \mathbb{E}\left[(m_{\delta_k}(\lambda))^2\right], \tag{73}$$

and the *Borel-Cantelli lemma* implies $\mathbb{P}\left[M^*_{\delta_k} \geq \frac{1}{k} \text{ i.o. }\right] = 0$. On the null set $\left\{M^*_{\delta_k} \geq \frac{1}{k} \text{ i.o. }\right\}$, we redefine $M_\delta(\cdot) \equiv 0$ for all δ, such that equation 68 still holds and now,

$$\forall \ \omega \in \Omega \ \exists K(\omega) \text{ such that } M^*_{\delta_k}(\omega) < \frac{1}{k} \ \forall k \geq K(\omega). \tag{74}$$

For each $t \in [0,T] \cap \mathbb{Q}$, the martingale $\{\mathbb{E}[\lambda(t)|\mathcal{F}(s)]\}_{0\leq s\leq T}$ is a martingale with respect to a *Brownian filtration*, whence it admits a stochastic integral representation. Therefore it has a continuous modification, which we call $\psi(s,t), \ 0 \leq s \leq T$, i.e., for each $s \in [0,T]$

$$\psi(s,t) = \mathbb{E}[\lambda(t)|\mathcal{F}(s)] \text{ a.s.}, \tag{75}$$

and without loss of generality, we take *every* path of $s \mapsto \psi(s,t)$ to be continuous, nonnegative and bounded above by C. In equations 68 and 75 the right-hand sides are defined only up to $\mathbb{P}$-a.s. equivalence, whereas the left-hand sides are defined for every $\omega \in \Omega$. More properly stated, a conditional expectation is an equivalence class of random variables, where the equivalence relation is almost

sure equality, and in equations 68 and 75 we are using the *Axiom of Choice* to choose representatives from equivalence classes.
We next study the dependence of $\psi(s,t)$ on t: If $0 \le t_1 < t_2 \le T$ and $t_1, t_2 \in \mathbb{Q}$, then

$$\psi(s,t_1) = \mathbb{E}[\lambda(t_1)|\mathcal{F}(s)] \le \mathbb{E}[\lambda(t_2)|\mathcal{F}(s)] = \psi(s,t_2) \text{ a.s.} \tag{76}$$

The null set $N_2(s,t_1,t_2)$ where this inequality fails can depend on s, t_1 and t_2. Define

$$N_2 \triangleq \bigcup_{s,t_1<t_2 \in [0,T]\cap\mathbb{Q}} N_2(s,t_1,t_2). \tag{77}$$

For $\omega \notin N_2$ we have

$$\psi(s,t_1) \le \psi(s,t_2) \ \forall \ s,t_1,t_2 \in [0,T] \cap \mathbb{Q}, \ t_1 < t_2. \tag{78}$$

If $s \in [0,T)$, we can choose a sequence $\{s_n\}_n$ in $[s,T] \cap \mathbb{Q}$ converging down to s. Conclude that for $\omega \notin N_2$,

$$\psi(s,t_1) = \lim_{n\to\infty} \psi(s_n,t_1) \le \lim_{n\to\infty} \psi(s_n,t_2) = \psi(s,t_2) \tag{79}$$

for all $s \in [0,T], \ t_1,t_2 \in [0,T] \cap \mathbb{Q}, t_1 < t_2$. For $\omega \in N_2$ we change the definition of $\psi(s,t)$, making it identically zero. Thus modified, $\psi(s,t)$ has all the properties it had before, and moreover, for *every* ω,

$$\psi(s,t_1) \le \psi(s,t_2) \ \forall \ s \in [0,T], \ t_1,t_2 \in [0,T] \cap \mathbb{Q}, \ t_1 < t_2. \tag{80}$$

For $s,t_1,t_2 \in [0,T] \cap \mathbb{Q}, \ t_1 < t_2$ we also have

$$\begin{aligned} \psi(s,t_2) - \psi(s,t_1) &= \mathbb{E}\left[\lambda(t_2) - \lambda(t_1)|\mathcal{F}(s)\right] \\ &\le \mathbb{E}\left[m_{t_2-t_1}|\mathcal{F}(s)\right] \\ &= M_{t_2-t_1}(s) \\ &\le M^*_{t_2-t_1} \text{ a.s.} \end{aligned} \tag{81}$$

The null set $N_3(s,t_1,t_2)$ where this inequality fails can depend on s, t_1 and t_2. Define

$$N_3 \triangleq \bigcup_{s,t_1<t_2 \in [0,T]\cap\mathbb{Q}} N_2(s,t_1,t_2). \tag{82}$$

For $\omega \notin N_3$ we have

$$\psi(s,t_2) - \psi(s,t_1) \le M^*_{t_2-t_1} \ \forall \ s,t_1,t_2 \in [0,T] \cap \mathbb{Q}, \ t_1 < t_2. \tag{83}$$

Using the continuity of $\psi(s,t)$ in s, we get this inequality for every $\omega \notin N_3$ and $s \in [0,T], \ t_1,t_2 \in [0,T] \cap \mathbb{Q}, \ t_1 < t_2$. For $\omega \in N_3$, we modify the definition of $\psi(s,t)$, making it identically zero. Thus modified, ψ has all the properties already established, and moreover, for *every* ω,

$$\psi(s,t_2) - \psi(s,t_1) \le M^*_{t_2-t_1} \ \forall \ s \in [0,T], \ t_1,t_2 \in [0,T] \cap \mathbb{Q}, \ t_1 < t_2. \tag{84}$$

An immediate consequence of equations 70, 74 and 84 is that for each fixed $s \in [0,T]$, the mapping $t \mapsto \psi(s,t)$ is right-continuous for all $t \in [0,T] \cap \mathbb{Q}$. This mapping $t \mapsto \psi(s,t)$ also has left-limits (again for $t \in [0,T] \cap \mathbb{Q}$ only), because of the nondecreasing property 80.

We need to define $\psi(s,t)$ when t is irrational. To do that, we fix $s \in [0,T]$ and an irrational number $t \in [0,T)$. We let $t_n \downarrow t$, where $t_n \in \mathbb{Q}$ for every n. For $m \geq n$, we have $t < t_m < t_n$ and for every ω,

$$0 \leq \psi(s,t_n) - \psi(s,t_m) \leq M^*_{t_n - t_m}. \tag{85}$$

From equations 70 and 74 we see that $\{\psi(s,t_n)\}_n$ is Cauchy, and thus has a limit. If $\{t'_n\}_n$ is a different sequence in $(t,T] \cap \mathbb{Q}$ converging down to t, then $\{\psi(s,t'_n)\}_n$ also has a limit, and so does the sequence of ψ-values obtained by interspersing the two sequences $\{t_n\}_n$ and $\{t'_n\}_n$. To avoid a contradiction, all three limits must be the same. Therefore,

$$\psi(s,t) \triangleq \lim_{t' \downarrow t, t' \in [0,T] \cap \mathbb{Q}} \psi(s,t') \tag{86}$$

is defined. For *every* ω, the mapping $t \mapsto \psi(s,t)$ is right-continuous for every $s \in [0,T]$. The nondecreasing property 80 is easily upgraded to

$$\psi(s,t_1) \leq \psi(s,t_2) \ \forall \ s, t_1, t_2 \in [0,T], \ t_1 < t_2, \ \forall \omega \in \Omega. \tag{87}$$

We know that for every $t \in [0,T] \cap \mathbb{Q}$ and every $\omega \in \Omega$ the mapping $s \mapsto \psi(s,t)$ is continuous. We want to establish this property when t is not necessarily rational. Let $\epsilon > 0$ be given. Let $\omega \in \Omega$ be given and choose k so that $M^*_{\delta_k} < \epsilon$ (see equation 80). If $t_1, t_2 \in [0,T] \cap \mathbb{Q}$ and $|t_2 - t_1| < \delta_k$ then equations 70 and 84 imply

$$|\psi(s,t_2) - \psi(s,t_1)| < \epsilon. \tag{88}$$

If $t \in [0,T]$ and $t_2 \in [0,T] \cap \mathbb{Q}$ satisfies $0 < t_2 - t < \delta_k$, then we can construct a sequence $t_1^{(n)} \downarrow t$ of rational numbers. For each n we have

$$0 \leq \psi(s,t_2) - \psi(s,t_1^{(n)}) < \epsilon, \tag{89}$$

and so

$$0 \leq \psi(s,t_2) - \psi(s,t) \leq \epsilon \ \forall \ s \in [0,T]. \tag{90}$$

Now let $s, s' \in [0,T]$ be given. We have

$$\begin{aligned} \psi(s',t) &= [\psi(s',t) - \psi(s',t_2)] + [\psi(s',t_2) - \psi(s,t_2)] \\ &\quad + [\psi(s,t_2) - \psi(s,t)] + \psi(s,t) \\ &\leq 0 + [\psi(s',t_2) - \psi(s,t_2)] + \epsilon + \psi(s,t) \\ &= [\psi(s',t_2) - \psi(s,t_2)] + \epsilon + \psi(s,t) \end{aligned} \tag{91}$$

and similarly

$$\psi(s',t) \geq [\psi(s',t_2) - \psi(s,t_2)] - \epsilon + \psi(s,t) \tag{92}$$

Letting $s' \to s$ in both 91 and 92 and using the continuity of $s' \mapsto \psi(t_2, s')$, we obtain

$$\begin{aligned} -\epsilon + \psi(s,t) &\leq \liminf_{s' \to s} \psi(s',t) \\ &\leq \limsup_{s' \to s} \psi(s',t) \\ &\leq \epsilon + \psi(s,t) \end{aligned} \tag{93}$$

Since ϵ is arbitrary, we must have

$$\psi(s,t) = \lim_{s' \to s} \psi(s', t). \tag{94}$$

In conclusion, $\psi(s,t)$, $0 \le s, t \le T$, satisfies for each $\omega \in \Omega$

(a) $s \mapsto \psi(s,t)$ is continuous for all $t \in [0,T]$,

(b) $t \mapsto \psi(s,t)$ is RCLL and nondecreasing for all $s \in [0,T]$

(c) $0 \le \psi(s,t) \le C$ for all $s, t \in [0,T]$

and for every $s, t \in [0,T]$ and a.e. ω

(d) $\psi(s,t) = \mathbb{E}[\lambda(t)|\mathcal{F}(s)]$.

For $s \in [0,T]$ and $t \in [0,T] \cap \mathbb{Q}$, (d) is just equation 75. For $t \in [0,T]$ we get (d) from equation 75 and the right-continuity of both the mappings $t \mapsto \psi(s,t)$ and $t \mapsto \lambda(t)$. This completes the proof of the lemma.

Corollary 4.8 *If $F^* = F_*$, then $u^*(S_0)$ is the upper hedging price. The condition $F^* = F_*$ is satisfied for a large class of options, including all path-independent options, options which depend continuously on the final stock-price, the maximal stock-price, the minimal stock-price and the average stock-price.*

4.8 Properties of the Value Functions

Theorem 4.9 *If F is lower semicontinuous, then all of the value functions $u(S_0;\lambda)$, $u_*(S_0;\lambda)$, $u_*(S_0)$, $u(S_0;\mathrm{c})$, $u(S_0;\mathrm{ac})$, $u(S_0;\mathrm{acb})$ are lower semicontinuous in the initial stock price S_0.*

Proof. We first establish the lower semicontinuity of $u(S_0;\lambda)$: Let $\{x_n\}_n$ be a sequence of nonnegative real numbers converging to S_0. Then

$$\begin{aligned}
\liminf_{n\to\infty} u(x_n;\lambda) &= \liminf_{n\to\infty} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} g\left(x_n e^{\sigma W+\mu-\lambda}\right)\right] \\
&\overset{\text{Fatou}}{\ge} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} \liminf_{n\to\infty} g\left(x_n e^{\sigma W+\mu-\lambda}\right)\right] \\
&\overset{g\ \text{lsc}}{\ge} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} g\left(S_0 e^{\sigma W+\mu-\lambda}\right)\right] \\
&= u(S_0;\lambda)
\end{aligned}$$

Since $F_*(\omega,\lambda) = F(\omega - \frac{1}{\sigma}\lambda)$, $u_*(S_0;\lambda) = u(S_0;\mathrm{c})$ and hence $u_*(S_0;\lambda)$ is lower semicontinuous in the initial stock price S_0. Next we consider $u(S_0;\mathrm{c})$: To show that this is lower semicontinuous, we can verify that for each real number a the set $\{x : u(x;\mathrm{c}) \le a\}$ is a closed subet of $\mathbb{R}$. This set can be written as an intersection of closed sets

$$\{x : u(x;\mathrm{c}) \le a\} = \bigcap_{\lambda \in \mathcal{L}_+^{\mathrm{c}}} \{x : u(x;\lambda) \le a\}$$

and each of the sets $\{x : u(x;\lambda) \le a\}$ is closed, because $u(S_0;\lambda)$ is lower semicontiunous. By the Extension Theorems we can now also say that $u_*(S_0)$, $u(S_0;\mathrm{ac})$ and $u(S_0;\mathrm{acb})$ are lower semicontinuous in the initial stock price S_0, because they all agree with $u(S_0;\mathrm{c})$. The proof is complete.

4.9 Results for Path-Independent Contingent Claims

Let $F = \phi(S_T)$ in this subsection. We want to establish properties of the face-lifted

$$\hat{\phi}(x) = \sup_{\lambda \geq 0} h_\lambda(x),$$

where $h_\lambda(x) \triangleq e^{-\alpha\lambda}\phi(xe^{-\lambda})$. The first result is

Theorem 4.10 *If ϕ is lower semicontinuous, then $\hat{\phi}$ is also lower semicontinuous.*

Proof. Let $\{x_n\}_n$ be a sequence of nonnegative real numbers converging to x. Then

$$\begin{aligned} \liminf_{n\to\infty} h_\lambda(x_n) &= e^{-\alpha\lambda} \liminf_{n\to\infty} \phi(x_n e^{-\lambda}) \\ &\geq e^{-\alpha\lambda}\phi(xe^{-\lambda}) \\ &= h_\lambda(x), \end{aligned}$$

because ϕ is lower semicontinuous. This implies that for any real number a the set $\{x : h_\lambda(x) \leq a\}$ is closed. Therefore

$$\{x : \hat{\phi}(x) \leq a\} = \bigcap_{\lambda \geq 0} \{x : h_\lambda(x) \leq a\}$$

is closed as well, which in turn implies the lower semicontinuity of $\hat{\phi}$.

If ϕ is not lower semicontinuous, then face-lifting does not necessarily produce the correct value function (see our example of the Cactus Option). On the other hand, we would really like to take an upper semicontinuous ϕ, because then we get the following

Theorem 4.11 *If ϕ is upper semicontinuous, then there exists for each $x \in [0,\infty)$ a maximizing number $\lambda^*(x) \in [0,\infty]$ such that*

$$\hat{\phi}(x) = e^{-\alpha\lambda^*(x)}\phi(xe^{-\lambda^*(x)}).$$

Moreover, the maximizing control $\lambda^* \in \mathcal{L}_+^{rc}$ for the value function $u^*(S_0)$ is given by the formula

$$\lambda^*(\omega, t) = \mathbb{I}_{\{t=T\}}\lambda^*(S(\omega)(T)).$$

Proof. Recall that every upper semicontinuous function on a compact domain attains its maximum and notice that we can write

$$\hat{\phi}(x) = \sup_{\lambda\in[0,\infty]} e^{-\alpha\lambda}\phi(xe^{-\lambda}) = \sup_{\nu\in[0,1]} \nu^\alpha\phi(x\nu).$$

Now, $[0,1]$ is compact, $\nu^\alpha\phi(x\nu)$ is upper semicontinuous in ν for each fixed x. Hence there exists indeed for each fixed x a maximizing number $\nu^*(x)$. Set $\lambda^*(x) = -\log\nu^*(x)$ to get the maximizing $\lambda^*(x)$. For the second part of the theorem, observe that due to the F^*/F_*-Identification theorem we know that

$$u^*(S_0) = \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)}\phi(S_T e^{-\lambda(T)})\right].$$

It should be pointed out that any maximizing control will be zero before time T, because the term inside the expectation only depends on $\lambda(T)$. Since jumps are allowed, it can save all of the necessary controlling effort until time T. This means that we are really only maximizing over random variables $\lambda(T) \geq 0$ rather than entire processes:

$$
\begin{aligned}
u^*(S_0) &= \sup_{\lambda(T)\geq 0} I\!E\left[e^{-rT-\alpha\lambda(T)}\phi(S_T e^{-\lambda(T)})\right] \\
&= \sup_{\lambda\geq 0} I\!E\left[e^{-rT-\alpha\lambda}\phi(S_T e^{-\lambda})\right] \\
&= I\!E\left[\sup_{\lambda\geq 0} e^{-rT-\alpha\lambda}\phi(S_T e^{-\lambda})\right].
\end{aligned}
$$

The last step just says that we can do the maximization pathwise. The proof of the theorem is complete.

The relevance of this theorem is the conjected one-to-one correspondence between the maximizing control and the cheapest hedge. The fact that it is true in the path-independent case underlines our conjecture. The next theorem dissolves the dilemma, that on the one hand for existence of the maximizing control we need upper semicontinuity, and on the other hand, to get the upper hedging price right, it would be nice to have lower semicontinuity of the payoff. Certainly, we are in good shape if the payoff is continuous, but we can say more:

Theorem 4.12 *Let the lower semicontinuous version of a given payoff ϕ be*

$$\phi_*(x) \triangleq \inf_{x_n\to x} \liminf_{n\to\infty} \phi(x_n).$$

(This is *the function F_*!) If*

$$\hat{\phi} = \hat{\phi}_*,$$

then even starting with the not necessarily lower semicontinuous ϕ face-lifting produces the correct upper hedging price.
If, additionally, ϕ is upper semicontinuous, then the control problem admits a maximizer.

Proof. Let $h(\phi)$ be the upper hedging price starting with the payoff ϕ. Since ϕ_* is lower semicontinuous, the results from [KARATZAS and SHREVE 5.7.1] tell us that

$$h(\phi_*) = I\!E\left[e^{-rT}\hat{\phi}_*(S_T)\right].$$

We continue as follows:

$$
\begin{aligned}
h(\phi_*) &= \mathbb{E}\left[e^{-rT}\hat{\phi}_*(S_T)\right] \\
&= \mathbb{E}\left[e^{-rT}\hat{\phi}(S_T)\right] \\
&= \mathbb{E}\left[\sup_{\lambda\geq 0} e^{-rT-\alpha\lambda}\phi(S_T e^{-\lambda})\right] \\
&= \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)}\phi(S_T e^{-\lambda(T)})\right] \\
&\geq \sup_{\lambda\in\mathcal{L}_+^{\mathrm{acb}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)}\phi(S_T e^{-\lambda(T)})\right] \\
&= h(\phi) \\
&\overset{\phi\geq\phi_*}{\geq} h(\phi_*),
\end{aligned}
$$

and so everything is in fact equal. This completes the proof of the theorem. Its relevance is of practical nature: The two digital put options

(1) $\phi(x) = \mathbb{I}_{\{x<B\}}$ (this is lower semicontinuous.)

(2) $\phi(x) = \mathbb{I}_{\{x\leq B\}}$ (this is upper semicontinuous.)

should have the same $\hat{\phi}$, and in fact they do! The first one is the lower semicontinuous version of the second one. However, only the upper semicontinuous version admits a maximizing control.

4.10 Future Research Topics

Conjecture 4.13 *There is a maximizing control* $\lambda^* \in \mathcal{L}_+^{rc}$ *for the value function* $u^*(S_0)$*, i.e.* $u^*(S_0) = u^*(S_0;\lambda^*)$.

Remark. There are some existence results available in the literature, for instance in [KARATZAS and SHREVE: Connections between Optimal Stopping and Singular Stochastic Control I. Monotone Follower Problems (1984)]. The existence for a minimizer there firmly rests on the convexity of the cost function. Our cost functions, however, completely fail to be concave. The most promising procedure here to prove existence of a maximizer is to establish the following result and then use the known fact that the cheapest hedge exists.

Conjecture 4.14 *There is a one-to-one correspondence between the optimal control (in case of its existence) and the cheapest hedge which superreplicates* F *and obeys the shortselling constraint.*

Remark. There is a striking similarity between the consumption process and the maximizing control: Both are adapted, nondecreasing, right-continuous processes starting at zero. They are, however, not immediately comparable: the control is dimensionless, but the consumption is a dollar-amount. Besides that, establishing the proposed one-to-one correspondence probably requires taking equivalence classes of controls, because we have learned in the example of a put option with a continuum of maximizing controls that maximizers themselves are not uniquely determined.

We want to outline how this correspondence works for the path-independent case. It has been proved in [KARATZAS and SHREVE, Methods of Mathematical Finance, chapter 5.7] that the hedge consists of the pair $(\pi(t), C(t))$ of portfolio-investment and consumption, which does the following: π is the Delta of the face-lifted option, and the consumption is given by

$$C(t) = \mathbb{I}_{\{t=T\}}(\hat{\phi}(S_T) - \phi(S_T)).$$

At time T, we skip the time-dependence and focus on the dependence of the final stock price S_T. Naming this x we can write

$$C(x) = \hat{\phi}(x) - \phi(x).$$

We are now instantly able to write down the equation that connects the consumption C and the optimal control λ^*: both of them are zero before T, and at time T, we must have for all $x \in [0, \infty)$

$$c(x) = e^{-\alpha\lambda^*(x)}\phi\left(xe^{-\lambda^*(x)}\right) - \phi(x). \tag{95}$$

This shows how easy it is to get the consumption if the maximizing control is known. The harder question is how to retrieve the maximizing control if only the consumption is given (and of course the contingent claim ϕ is known). We believe this can be done as well, and here is how: We take the assumption of theorem 4.12: An upper semicontinuous ϕ which produces the same $\hat{\phi}$ as its lower semicontinuous version ϕ_*. We have proved in theorem 4.10 that $\hat{\phi}$ is lower semicontinuous. Therefore,

$$\hat{\phi} - \phi \text{ is lower semicontinuous,} \tag{96}$$

and consequently, the set

$$I \triangleq \{x : C(x) > 0\} \text{ is open.} \tag{97}$$

We may thus write this set as a disjoint union of countably many open intervals

$$\{x : C(x) > 0\} = \dot{\bigcup_{n\in\mathbb{N}}} (l_n, r_n). \tag{98}$$

We allow one of the right endpoints r_n to be ∞, and we allow I to be empty (for those options which do not cause a shortselling problem). The maximizing control λ^* is then given by the formula

$$\lambda^*(x) \triangleq \sum_{n\in\mathbb{N}} \mathbb{I}_{(l_n,r_n)}(x)\left[\log(x) - \log(l_n)\right]. \tag{99}$$

For a proof, we must verify that equation 95 holds. For $x \notin I$ this equation just says $0 = \phi(x) - \phi(x)$, which is a good sign. For $x \in I$, x must be contained in one of the intervals (l_n, r_n), and in this case equation 95 looks like

$$\hat{\phi}(x) = \phi(l_n)\left(\frac{l_n}{x}\right)^{\alpha}. \tag{100}$$

This is the solution to the ordinary differential equation

$$\alpha\hat{\phi}(x) + x\hat{\phi}'(x) = 0, \quad x \in (l_n, r_n) \tag{101}$$
$$\hat{\phi}(l_n) = \phi(l_n). \tag{102}$$

The rest is a verification of conjecture 4.17, which says that for positive consumption the shortselling constraint is tight. This would imply equation 101. The initial condition 102 is obvious, because $\hat{\phi}$ is certainly never below ϕ, and $\hat{\phi}(l_n) > \phi(l_n)$ would violate the minimality.

The whole procedure probably becomes much clearer, when we look at an example. Let's take the vanilla put option $\phi(x) = (K-x)^+$. We get (see example section)

$$\hat{\phi}(x) = \left\{ \begin{array}{ll} K - x & \text{if } x \leq \frac{\alpha}{1+\alpha}K \\ \frac{K}{1+\alpha}(\frac{\alpha K}{(1+\alpha)x})^\alpha & \text{if } x \geq \frac{\alpha}{1+\alpha}K \end{array} \right\} \tag{103}$$

and so

$$I = \{x : C(x) > 0\} = \left(\frac{\alpha}{1+\alpha}K, \infty\right). \tag{104}$$

Equation 99 translates into

$$\begin{aligned} \lambda^*(x) &= I\!I_{\{x > \frac{\alpha}{1+\alpha}K\}} \left[\log(x) - \log\left(\frac{\alpha}{1+\alpha}K\right)\right] \\ &= \left[\log(x) - \log\left(\frac{\alpha}{1+\alpha}K\right)\right]^+ \end{aligned} \tag{105}$$

as it should and setting $l_1 \triangleq \frac{\alpha}{1+\alpha}K$ we can write $\hat{\phi}$ as

$$\hat{\phi}(x) = \left\{ \begin{array}{ll} \phi(x) & \text{if } x \leq l_1 \\ \phi(l_1)\left(\frac{l_1}{x}\right)^\alpha & \text{if } x \geq l_1 \end{array} \right\} \tag{106}$$

This example visualizes the problem in a wonderful way: Whereas C describes the *horizontal* distance of $\hat{\phi}$ and ϕ, the optimal control λ^* describes the *vertical logarithmic* distance of $\hat{\phi}$ and $\check{\phi}$, where the *dropped option* $\check{\phi}$ is the smallest nonnegative option which produces the same $\hat{\phi}$ as ϕ. In our example

$$\check{\phi}(x) = \left\{ \begin{array}{ll} K - x & \text{if } x \leq l_1 \\ 0 & \text{if } x > l_1 \end{array} \right\} \tag{107}$$

By drawing a diagram, the reader will notice that we have given a *graphical algorithm* to determine the maximizing control λ^* purely from knowing the consumption C!

How does equation 95 look in the path-dependent case? Let $F = g(S)$ be an upper semicontinuous path-dependent contingent claim and assume that its extension F^* equals g. Then general option pricing theory tells us that any candidate for a maximizing control λ^* must satisfy

$$I\!E\left[\int_0^T e^{-rt}\,dC(t) + e^{-rT}g(S)\right] = I\!E\left[e^{-rT-\alpha\lambda^*(T)}g\left(Se^{-\lambda^*}\right)\right] \tag{108}$$

In the path-independent setup this is actually an ω-wise agreement. We would thus pose the question: Is equation 108 true even before we take expectations? And would it help to retrieve λ^* from a known g and C?

Conjecture 4.15 *$u^*(S_0)$ is the upper semicontinuous version of $u_*(S_0)$, as our notation already indicates.*

We are motivated to state this conjecture essentially from the Up-and-Out Call. The path-independent case certainly has this property, but since the Heat-equation smoothens everything out, we only see the statement happen at expiration.

Conjecture 4.16 *The shortselling constraint is tight (i.e. satisfied with equality) if and only if the optimal control is active.*

In the path-independent case we can clarify this as follows and attempt the

Conjecture 4.17 *Assuming enough differentiability, if $\hat{\phi}(x) > \phi(x)$ then $\hat{\phi}$ satisfies the shortselling constraint with equality, i.e. $w(x) \triangleq \alpha\hat{\phi}(x) + x\hat{\phi}'(x) = 0$.*

Without loss of generality we let $\hat{\phi}(x) > \phi(x)$ for x in one of the intervals (l_n, r_n). We must then show that for these x, the function $w(x) = 0$. If the maximizer λ^* is also differentiable, then we get

$$w(x) = e^{-\alpha\lambda^*(x)} \left(1 - x\frac{\partial\lambda^*}{\partial x}(x)\right) \left[\alpha\phi\left(xe^{-\lambda^*(x)}\right) + xe^{-\lambda^*(x)}\phi'\left(xe^{-\lambda^*(x)}\right)\right] \tag{109}$$

and this is indeed a zero, if we can derive the particular form of λ^* which we have outlined above, namely $\lambda^*(x) = \log(x) - \log(l_n)$. Here $\frac{\partial\lambda^*}{\partial x}(x) = \frac{1}{x}$ causing the term $\left(1 - x\frac{\partial\lambda^*}{\partial x}(x)\right)$ to be zero. Additionally, for this λ^* the other term collapses to $[\alpha\phi(l_n) + l_n\phi'(l_n)]$. The left endpoint l_n, left of which this quantity must be nonnegative (otherwise l_n would be further to the left), and right of which this quantity must be negative (otherwise we wouldn't have lifted ϕ there), makes the term (assuming enough continuity) $[\alpha\phi(l_n) + l_n\phi'(l_n)]$ equal to zero.
This conjecture gives us a good understanding how to produce $\hat{\phi}$ in an algorithmic way: Check for all $x \geq 0$ from left to right, whether $\phi(x)$ satisfies the shortselling constraint $[\alpha\phi(x) + x\phi'(x)] \geq 0$ (A subdifferential interpretation of ϕ' is sufficient). Take l_1 be the "first" (i.e. the infimum) x, where the shortselling constraint is violated, set $\hat{\phi}(x) = \phi(x)$ for all $x \leq l_1$ and from l_1 onwards take $\hat{\phi}(x) = \phi(l_1)\left(\frac{l_1}{x}\right)^\alpha$. Do this until r_1 being the first x when $\hat{\phi}(x)$ hits $\phi(x)$ again. After r_1 set $\hat{\phi}(x) = \phi(x)$ and let l_2 be the first $x > r_1$ where $\phi(x)$ violates the shortselling constraint, etc. This works beautifully for the vanilla put option and produces $\hat{\phi}$ in a very *transparent and quick* way. Check!! It completely illuminates the general structure of face-lifting as well as the structure of a maximizing control: *$\hat{\phi}$ agrees with ϕ where ϕ already satisfies the shortselling constraint and thus requires no further lifting. Otherwise $\hat{\phi}$ is the decreasing branch of a hyberbola, whose left endpoint is continuous and there may be at most a jump up at the right endpoint. λ^* pushes the final stock price back to these left endpoints in all the hyperbolic regions.* Everything has been visualized!

In generaI I can only observe that these conjectures are correct in the examples. I strongly believe that they are true in some generality at least under certain conditions. However, I will postpone the clarifiction of these conjectures to future research.

5 Examples

5.1 The Digital Put Option

$$F(\omega) \triangleq \phi\left(S_0 e^{\sigma\omega(T)+(r-\frac{1}{2}\sigma^2)T}\right), \quad \phi(x) \triangleq C I\!I_{\{x\le B\}}$$

for some positive nominal amount C and positive strike B. We may frequently use the shorthand notation $F = \phi(S_T)$ to suppress the little ω's. Then

$$F^* = \phi(S_T e^{-\lambda(T)})$$

and the control problem becomes

$$\begin{aligned} u(S_0) &= \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} \int_\Omega e^{-rT-\alpha\lambda(T,\omega)} F^*(\omega,\lambda(\omega))\, dI\!P(\omega) \\ &= e^{-rT} \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} I\!E[e^{-\alpha\lambda(T)}\phi(S_T e^{-\lambda(T)})] \\ &= e^{-rT} I\!E[e^{-\alpha\lambda^*(T)}\phi(S_T e^{-\lambda^*(T)})], \end{aligned}$$

where

$$\lambda^*(t) = (\log S_T - \log B)^+ I\!I_{\{t=T\}} \tag{110}$$

is the maximizing control: a process that is identically zero before expiration time T and has a random jump up at T. This means that the stock price remains uncontrolled until time T, and at T, the stock is pushed down to the strike, if it is above the strike, and otherwise remains uncontrolled. To understand that this is true, recall that ϕ can only take values in $\{0, C\}$, so to maximize the above expected value, it is always (i.e. for all $I\!P$-a.e. ω) better to get a C than a 0, no matter how much that C will be discounted. Now, to make sure $\phi = C$ for $I\!P$-a.e. ω, S_T must stay at or below the strike B. Since unnecessary discounting should be avoided, λ only pushes, when S_T is above B. This explains why the formula is right at the final time T. To see why λ does not act any time before, recall that all controls are nondecreasing: this means that once we have pushed there is now way to take this effort back, however, pushing at any time causes discounting and this effect reduces the value. But for any fixed path ω, when we have pushed at a time prior to time T, it could be, that *at* time T, we need not have pushed at all, because the stock is already below the strike.
With this λ^*, $u(S_0)$ can be written as

$$u(S_0) = e^{-rT} I\!E[\hat{\phi}(S_T)].$$

We used the *face-lifting equation*

$$\hat{\phi}(x) \triangleq \sup_{\pi\in\tilde{C}} [e^{-\delta(\nu)}\phi(xe^{-\nu})] \overset{\text{here}}{=} \left\{ \begin{array}{ll} 1 & \text{if } x \le B \\ (\frac{B}{x})^\alpha & \text{if } x \ge B \end{array} \right\},$$

where the *support function* of the closed convex set $C \stackrel{\text{here}}{=} [-\alpha, \infty)$ is

$$\delta(\nu) \triangleq \sup_{\nu \in C}(-\pi\nu) \stackrel{\text{here}}{=} \left\{ \begin{array}{ll} \alpha\nu & \text{if } \nu \geq 0 \\ \infty & \text{if } \nu < 0 \end{array} \right\}$$

and

$$\tilde{C} \triangleq \{\nu : \delta(\nu) < \infty\} \stackrel{\text{here}}{=} [0, \infty)$$

is its *effective domain* [see ROCKAFELLAR]. This shows that our approach contains the results of the paper by [BROADIE, CVITANIĆ and SONER] as a special case. Their maximizing ν of the *face-lifting equation* is our maximizing $\lambda(T)$ of the stochastic control problem. This correspondence is true for all non-pathdependent options. For such options we can always repeat the argument above to conclude that $\lambda(t) = 0$ for $t < T$. In our case we could compute $u(S_0; 0)$ and $u^*(S_0)$ explicitely. The maximizing λ^* does not depend on the values ϕ takes below B, as long as ϕ is nondecreasing before B.
The *generalized face-lifting equation* can be described as

$$\hat{F}(\omega) = e^{-\alpha\lambda^*(\omega)(T)} F^*(\omega, \lambda^*(\omega)).$$

This means that, suppose we have found a maximizing control process λ^*, and we compute $\hat{F}$ as proposed above, then $\hat{F}$ is the option we are actually hedging. By doing that, we will superreplicate the payoff F and our hedge will satisfy the shortselling constraint. Moreover, the initial value $\mathbb{E}[e^{-rT}\hat{F}]$ is the minimal amount to do that.
Any other non-pathdependent option is handled similarly and covered in the paper by [BROADIE, CVITANIĆ and SONER]. However, we need to point out at this place that the face-lifting procedure does not work for all functions, for instance:

5.2 A Cactus Option

$$F = \phi(S_T), \quad \phi(x) \triangleq \mathbb{I}_{\{x=K\}}$$

for some positive strike K. Then

$$F^* = \phi(S_T e^{-\lambda(T)})$$

and the control problem becomes

$$\begin{aligned} u(S_0) &= \sup_{\lambda \in \mathcal{L}_+^{\text{rc}}} \int_\Omega e^{-rT - \alpha\lambda(T,\omega)} F^*(\omega, \lambda(\omega))\, d\mathbb{P}(\omega) \\ &= e^{-rT} \sup_{\lambda \in \mathcal{L}_+^{\text{rc}}} \mathbb{E}[e^{-\alpha\lambda(T)} \phi(S_T e^{-\lambda(T)})] \\ &= e^{-rT} \mathbb{E}[e^{-\alpha\lambda^*(T)} \phi(S_T e^{-\lambda^*(T)})], \end{aligned}$$

where

$$\lambda^*(t) = (\log S_T - \log K)^+ \, \mathbb{I}_{\{t=T\}} \tag{111}$$

is the maximizing control. Using this λ^* we can compute the expected value as

$$\mathbb{E}[e^{-\alpha\lambda^*(T)} \phi(S_T e^{-\lambda^*(T)})] = \mathbb{E}[\hat{\phi}(S_T)],$$

where

$$\hat{\phi}(x) = \left\{ \begin{array}{ll} 0 & \text{if } x < B \\ (\frac{B}{x})^\alpha & \text{if } x \geq B \end{array} \right\}.$$

Of course, there must be something wrong. The cactus option ϕ is almost surely equal to zero, whence we would expect $\hat{\phi}$ to be zero as well. This example shows that we do need to require ϕ to be lower semicontinuous or at least $\hat{\phi} = \hat{\phi_*}$, where

$$\phi_*(x) \triangleq \inf\left\{ \liminf_{n\to\infty} \phi(x_n) \middle| x_n \overset{n\to\infty}{\longrightarrow} x \right\}$$

is the lower semicontinuous version of ϕ. Our cactus option satisfies none of these assumptions: ϕ is not lower semicontinuous and ϕ_* is identically zero and so is $\hat{\phi_*}$.

5.3 The Vanilla Put Option

Although the analysis of path-indedpendent options is now clear, we still want to look at a vanilla put option, because its analysis will help us to take further steps to the path dependent lookback put option, the Asian put option and the barrier put options. Let

$$F = \phi(S_T), \quad \phi(x) \triangleq (K - x)^+$$

for some positive strike K. Then

$$F^* = \phi(S_T e^{-\lambda(T)})$$

and the control problem becomes

$$\begin{aligned} u^*(S_0) &= \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} \int_\Omega e^{-rT-\alpha\lambda(T,\omega)} F^*(\omega, \lambda(\omega))\, dI\!P(\omega) \\ &= e^{-rT} \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} I\!E[e^{-\alpha\lambda(T)}\phi(S_T e^{-\lambda(T)})] \\ &= e^{-rT} I\!E[e^{-\alpha\lambda^*(T)}\phi(S_T e^{-\lambda^*(T)})], \end{aligned}$$

where

$$\lambda^*(t) = \left(\log S_T - \log\left[\frac{\alpha}{1+\alpha}K\right]\right)^+ I\!I_{\{t=T\}} \tag{112}$$

is the maximizing control. Using this λ^* we can compute the expected value as

$$I\!E[e^{-\alpha\lambda^*(T)}\phi(S_T e^{-\lambda^*(T)})] = I\!E[\hat{\phi}(S_T)],$$

where

$$\hat{\phi}(x) = \left\{ \begin{array}{ll} K - x & \text{if } x \leq \frac{\alpha}{1+\alpha}K \\ \frac{K}{1+\alpha}(\frac{\alpha K}{(1+\alpha)x})^\alpha & \text{if } x \geq \frac{\alpha}{1+\alpha}K \end{array} \right\}.$$

5.4 A Put Option with a continuum of maximizing controls

$$F = \phi(S_T), \quad \phi(x) \triangleq \left\{ \begin{array}{ll} (a-x)\alpha a^{-\alpha-1} + a^{-\alpha} & \text{if } 0 \leq x \leq a \\ x^{-\alpha} & \text{if } a \leq x \leq b \\ (b-x)\alpha b^{-\alpha-1} + b^{-\alpha} & \text{if } b \leq x \leq b\frac{1+\alpha}{\alpha} \\ 0 & \text{if } b\frac{1+\alpha}{\alpha} \leq x \end{array} \right.$$

for some numbers $0 < a < b$. Then

$$F^* = \phi(S_T e^{-\lambda(T)})$$

and as before the control problem is solved by

$$u^*(S_0) = e^{-rT} \mathbb{E}[e^{-\alpha\lambda^*(T)} \phi(S_T e^{-\lambda^*(T)})],$$

where $\lambda^*(t) = 0$ for $t < T$ and we may take *any*

$$\lambda^*(T) \in \left[(\log \frac{S_T}{b})^+, (\log \frac{S_T}{a})^+ \right] \tag{113}$$

as a maximizing control. Using one of them we can compute the expected value as

$$\mathbb{E}[e^{-\alpha\lambda^*(T)} \phi(S_T e^{-\lambda^*(T)})] = \mathbb{E}[\hat{\phi}(S_T)],$$

where

$$\hat{\phi}(x) = \left\{ \begin{array}{ll} (a-x)\alpha a^{-\alpha-1} + a^{-\alpha} & \text{if } 0 \leq x \leq a \\ x^{-\alpha} & \text{if } a \leq x \end{array} \right\}.$$

To get $\hat{\phi}$ from ϕ, recall that $\hat{\phi}$ is the smallest payoff which dominates ϕ and satisfies the shortselling constraint $\alpha\phi(x) + x\phi'(x) \geq 0$. Here, ϕ already satisfies this constraint for $x \leq b$. Above b, $\hat{\phi}$ satisfies the constraint with equality, so it can't be any smaller.
Such an option is not likely to be traded, but we learn from this example that a maximizing control is in general not uniquely determined.

5.5 The Lookback Put Option

This is an example which illuminates that, although controlling takes place only at the end, $\lambda(T)$ is not always just a function of $S(T)$, but can be a function of the whole path.

$$F = \max_{t\in[0,T]} S(t) - S(T)$$

F^* has been identified as

$$F^* = \sup_{t\in[0,T]} [S(t)e^{-\lambda(t)}] - S(T)e^{-\lambda(T)}$$

and the control problem becomes

$$u^*(S_0) = \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} \left(\sup_{t\in[0,T]} [S(t)e^{-\lambda(t)}] - S(T)e^{-\lambda(T)} \right) \right].$$

First observe pathwise that controlling at the end only is at least as good as splitting the controlling effort over time, whence we need to consider only controls satisfying $\lambda(t) = 0$ for $t < T$. Secondly observe that for such controls

$$\sup_{t\in[0,T]} [S(t)e^{-\lambda(t)}] = \sup_{t\in[0,T]} S(t) = \max_{t\in[0,T]} S(t).$$

Now the control problem reads as

$$u^*(S_0) = \sup_{\lambda \geq 0} \mathbb{E}\left[e^{-rT-\alpha\lambda} \left(\max_{t\in[0,T]} S(t) - S(T)e^{-\lambda} \right) \right], \tag{114}$$

which is exactly the same control problem as in the case of the vanilla put option. The only difference is that $\max_{t\in[0,T]} S(t)$ takes the role of the strike K. We can hence do the same computation to get the maximizing control

$$\lambda^*(t) = \left(\log S(T) - \log\left[\frac{\alpha}{1+\alpha}\max_{t\in[0,T]} S(t)\right]\right)^+ I\!I_{\{t=T\}}. \tag{115}$$

To compute the value function we need to evaluate

$$u^*(S_0) = e^{-rT} I\!E[\hat{\phi}(S_T, \max_{t\in[0,T]} S(t))],$$

where

$$\hat{\phi}(x,y) = \left\{ \begin{array}{ll} y - x & \text{if } x \leq \frac{\alpha}{1+\alpha} y \\ \frac{y}{1+\alpha}\left(\frac{\alpha y}{(1+\alpha)x}\right)^\alpha & \text{if } x \geq \frac{\alpha}{1+\alpha} y \end{array} \right\}.$$

The minimality of this value function is covered by the general theory. To see that the value function will allow a hedge that respects the shortselling constraint and superreplicates a lookback put option, we can actually go through a direct argument using partial differential equations. This is a worthwile exercise:

Proposition 5.1 *$\hat{\phi}(x,y)$ is increasing and convex in y.*

This can be verified by ordinary calculus. To get hold of a value function with payoff $\hat{\phi}(x,y)$, define the following partial differential equation: Assume $\hat{v}(t,x,y)$ satisfies for all $x < y$

1. $\hat{v}_t - r\hat{v} + rx\hat{v}_x + \frac{1}{2}\sigma^2 x^2 \hat{v}_{xx} = 0$
2. $\hat{v}(T,x,y) = \hat{\phi}(x,y)$
3. $\hat{v}_y(t,x,x) = 0$ (Neuman boundary condition)

Define

$$M_t \triangleq \max_{u\in[0,t]} S(u) \tag{116}$$

$$Y(t) \triangleq e^{-rT}\hat{v}(t, S(t), M(t)), \tag{117}$$

compute the differential dY, integrate from t to T and take expectations conditioned on $S(t) = x$, $M(t) = y$, resulting in

$$\hat{v}(t,x,y) = e^{-r(T-t)} I\!E[\hat{\phi}(S(T), M(T)) | S_t = x, M_t = y]. \tag{118}$$

This value function certainly dominates the corresponding value function of the unconstrained lookback put option, $v(t,x,y)$, which is the same as $\hat{v}$ except that $\hat{v}(T,x,y) = y - x \leq \hat{\phi}(x,y)$.

Proposition 5.2 *$\hat{v}(t,x,y)$ is convex in y.*

To prove this, let

$$M(t,T) \triangleq \max_{u\in[t,T]} S(u), \tag{119}$$

such that $M_T = \max(M_t, M(t,T))$. If now for some $\xi \in [0,1]$, $M_t = \xi y_1 + (1-\xi)y_2$, then

$$\begin{aligned} M_T &= \max[M(t,T), \xi y_1 + (1-\xi)y_2] && (120)\\ &\le \xi \max[(M(t,T), y_1] + (1-\xi)\max[M(t,T), y_2], && (121)\end{aligned}$$

because the function $(a,b) \mapsto \max[a,b]$ is convex in both variables. Now let $\xi \in [0,1]$ be given. We compute

$$\begin{aligned} & e^{-r(T-t)}\hat{v}(t,x,\xi y_1 + (1-\xi)y_2) \\ = \;& \mathbb{E}[\hat{\phi}(S_T, M_T)|S_t = x, M_t = \xi y_1 + (1-\xi)y_2] \\ = \;& \mathbb{E}[\hat{\phi}(S_T, \max[M(t,T), \xi y_1 + (1-\xi)y_2])|S_t = x] \\ \le \;& \mathbb{E}[\hat{\phi}(S_T, \xi\max[(M(t,T), y_1] + (1-\xi)\max[M(t,T), y_2])|S_t = x] \\ \le \;& \xi\mathbb{E}[\hat{\phi}(S_T, \max[M(t,T), y_1])|S_t = x] \\ & +(1-\xi)\mathbb{E}[\hat{\phi}(S_T, \max[M(t,T), y_2])|S_t = x] \\ = \;& e^{-r(T-t)}[\xi\hat{v}(t,x,y_1) + (1-\xi)\hat{v}(t,x,y_2)], \end{aligned}$$

where we have used that $\hat{\phi}(x,y)$ is increasing and convex in y. The convexity of $\hat{v}(t,x,y)$ in its y-variable follows and results in

$$\hat{v}_{yy}(t,x,y) \ge 0 \;\; \forall\, x \le y. \tag{122}$$

Proposition 5.3 *$\hat{v}$ respects the shortselling constraint.*

Now let us define

$$w(t,x,y) \triangleq \alpha\hat{v}(t,x,y) + x\hat{v}_x(t,x,y). \tag{123}$$

We derive

$$w_y(t,x,y) = \alpha\hat{v}_y(t,x,y) + x\hat{v}_{xy}(t,x,y). \tag{124}$$

We have assumed that $\hat{v}_y(t,x,x) = 0$, so taking its total differential yields $\hat{v}_{xy}(t,x,x) + \hat{v}_{yy}(t,x,x) = 0$ and thus $w_y(t,x,x) = -x\hat{v}_{yy}(t,x,x) \le 0$ for all x and t.
Here is a summary of the properties of w:

1. $w_t - rw + rxw_x + \frac{1}{2}\sigma^2 x^2 w_{xx} = 0$
2. $w(T,x,y) = (\alpha y - (1+\alpha)x)^+ \ge 0$
3. $w_y(t,x,x) = -x\hat{v}_{yy}(t,x,x) \le 0$

Now define

$$Y(t) \triangleq e^{-rT}w(t,S(t),M(t)), \tag{125}$$

compute its differential $\mathrm{d}Y$, integrate from t to T and take expectations conditioned on $S(t) = x$, $M(t) = y$, resulting in

$$w(t,x,y) \ge e^{-r(T-t)}\mathbb{E}[w(T,S(T),M(T))|S_t = x, M_t = y] \ge 0, \tag{126}$$

and consequently by the definition of w

$$\alpha\hat{v}(t,x,y) + x\hat{v}_x(t,x,y) \ge 0 \;\; \forall\, t \le T,\; x \le y, \tag{127}$$

which means that $\hat{v}$ respects the shortselling constraint.

5.6 The Asian Put Option

The control problem is easy to identify even here. Let

$$F = \left(\frac{1}{T} \int_0^T S(t)\, dt - S(T) \right)^+ \tag{128}$$

F^* has been identified as

$$F^* = \left(\frac{1}{T} \int_0^T S(t) e^{-\lambda(t)}\, dt - S(T) e^{-\lambda(T)} \right)^+ \tag{129}$$

and the control problem becomes

$$u^*(S_0) = \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} I\!\!E \left[e^{-rT - \alpha\lambda(T)} \left(\frac{1}{T} \int_0^T S(t) e^{-\lambda(t)}\, dt - S(T) e^{-\lambda(T)} \right)^+ \right]. \tag{130}$$

First observe pathwise that controlling at the end only is at least as good as splitting the controlling effort over time, whence we need to consider only controls satisfying $\lambda(t) = 0$ for $t < T$. Secondly observe that for such controls

$$\int_0^T S(t) e^{-\lambda(t)}\, dt = \int_0^T S(t)\, dt.$$

Now the control problem reads as

$$u^*(S_0) = \sup_{\lambda \geq 0} I\!\!E \left[e^{-rT - \alpha\lambda} \left(\frac{1}{T} \int_0^T S(t) - S(T) e^{-\lambda} \right)^+ \right], \tag{131}$$

which is exactly the same control problem as in the case of the vanilla put option. The only difference is that $\frac{1}{T} \int_0^T S(t)\, dt$ takes the role of the strike K. We can hence do the same computation to get the maximizing control

$$\lambda^*(t) = \left(\log S(T) - \log \left[\frac{\alpha}{1+\alpha} \frac{1}{T} \int_0^T S(t)\, dt \right] \right)^+ I\!\!I_{\{t=T\}}. \tag{132}$$

To compute the value function we need to evaluate

$$u^*(S_0) = e^{-rT} I\!\!E \left[\hat{\phi} \left(S_T, \frac{1}{T} \int_0^T S(t)\, dt \right) \right],$$

where

$$\hat{\phi}(x, y) = \left\{ \begin{array}{ll} y - x & \text{if } x \leq \frac{\alpha}{1+\alpha} y \\ \frac{y}{1+\alpha} \left(\frac{\alpha y}{(1+\alpha) x} \right)^{\alpha} & \text{if } x \geq \frac{\alpha}{1+\alpha} y \end{array} \right\}.$$

5.7 The Up and Out Call Option

$$F = (S(T) - K)^+ \mathbb{I}_{\{S(t) \leq B \forall t \in [0,T]\}}$$

for some positive strike K and positive barrier $B > K$.

$$F^* = (S(T)e^{-\lambda(T)} - K)^+ \mathbb{I}_{\{S(t)e^{-\lambda(t)} \leq B \forall t \in [0,T]\}}$$

The control problem

$$u^*(S_0) = \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} \mathbb{E}\left[e^{-rT - \alpha\lambda(T)} \left((S(T)e^{-\lambda(T)} - K)^+ \mathbb{I}_{\{S(t)e^{-\lambda(t)} \leq B \forall t \in [0,T]\}}\right)\right]$$

is solved by

$$\lambda^*(t) = \max_{u \in [0,t]} [\log S(u) - \log B]^+, \tag{133}$$

which is the singularly continuous process, which pushes the stock just enough to prevent it from crossing the barrier. If we were only going to check at the final time T, whether the barrier has been crossed, then λ^* would coincide with the optimal control of the Digital Put Option. We observe also that a *time dependent barrier* $B(t)$ would yield the similar solution

$$\lambda^*(t) = \max_{u \in [0,t]} [\log S(u) - \log B(u)]^+. \tag{134}$$

In this case we could still solve the value function numerically, although an explicit analytical solution like in the constant barrier case seems impossible.

5.8 The Realistic Up and Out Call Option

$$F = (S(T) - K)^+ \prod_{i=1}^{N} \mathbb{I}_{\{S(t_i) \leq B\}}$$

for some positive strike K, some positive barrier $B > K$ and some checkpoints $0 \leq t_1 < t_2 < \cdots < t_N \leq T$.

$$F^* = (S(T)e^{-\lambda(T)} - K)^+ \prod_{i=1}^{N} \mathbb{I}_{\{S(t_i)e^{-\lambda(t_i)} \leq B\}}$$

The control problem

$$u^*(S_0) = \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} \mathbb{E}\left[e^{-rT - \alpha\lambda(T)} \left((S(T)e^{-\lambda(T)} - K)^+ \prod_{i=1}^{N} \mathbb{I}_{\{S(t_i)e^{-\lambda(t_i)} \leq B\}}\right)\right]$$

is solved by

$$\lambda^*(t) = \max_{u \in [0,t] \cap \{t_1, \ldots, t_N\}} [\log S(u) - \log B]^+, \tag{135}$$

which is the singular jump process that pushes the stock down to the barrier at all the checkpoints if necessary. We note that here we could use a time dependent barrier as well.

5.9 The Up and Out Put Option

$$F = (K - S(T))^+ I\!I_{\{S(t) \le B \forall t \in [0,T]\}}$$

for some positive strike K and positive barrier B.

$$F^* = (K - S(T)e^{-\lambda(T)})^+ I\!I_{\{S(t)e^{-\lambda(t)} \le B \forall t \in [0,T]\}}$$

The control problem

$$u^*(S_0) = \sup_{\lambda \in \mathcal{L}_+^{\mathrm{rc}}} I\!E\left[e^{-rT-\alpha\lambda(T)}\left(K - (S(T)e^{-\lambda(T)})^+ I\!I_{\{S(t)e^{-\lambda(t)} \le B \forall t \in [0,T]\}}\right)\right]$$

is solved by

$$\begin{aligned} \lambda^*(t) &= \max_{u \in [0,t]} [\log S(u) - \log B(u)]^+, \qquad (136) \\ B(u) &= \left\{ \begin{array}{ll} B & \text{if } u < T \\ \min\left[B, \frac{\alpha}{1+\alpha} K\right] & \text{if } u = T \end{array} \right\}. \end{aligned}$$

which is the singular process, which pushes the stock just enough to prevent it from crossing the barrier. At the final time we are back at the vanilla put, so we give the stock an additional final push down to $\frac{\alpha}{1+\alpha}K$, if it is above that number. It is a worthwhile exercise to solve the control problem directly and compute $u^*(S_0)$ explicitely. As in the case of the Up-and-Out Call we define $v(t, x; \alpha)$ to be the solution to the partial differential equation with the conditions

$$\begin{aligned} -rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} &= 0, \quad 0 \le t < T, \quad 0 < x < B \\ \alpha v(t, B; \alpha) + Bv_x(t, B; \alpha) &= 0, \quad 0 \le t \le T \\ \hat{\phi}(x) \triangleq v(T, x; \alpha) &= \left\{ \begin{array}{ll} K - x & \text{if } x \le B \wedge \frac{\alpha}{1+\alpha}K \\ \frac{K}{1+\alpha}\left(\frac{\alpha K}{(1+\alpha)x}\right)^\alpha & \text{if } B \ge x \ge \frac{\alpha}{1+\alpha}K \\ 0 & \text{if } x > B \end{array} \right. \end{aligned}$$

To find the solution let

$$M(t) \triangleq \max_{0 \le u \le t} S(u). \qquad (137)$$

We define the value of an auxiliary contingent claim by

$$w(t, x; \alpha) \triangleq I\!E\left[e^{-r(T-t)}[\alpha K - (1+\alpha)S(T)]^+ I\!I_{\{M(T) < B\}} | S_t = x\right], \qquad (138)$$

and list some properties of $w(t, x; \alpha)$:

(i) $e^{-rt}w(t, S(t); \alpha)$ is a martingale, and therefore

(ii) $w(t, x; \alpha)$ satisfies the Black-Scholes partial differential equation.

(iii) $w(t, B; \alpha) = 0$

(iv) $0 \le w(t, x; \alpha) \le \alpha K$ and $w(t, 0; \alpha) = e^{-r(T-t)}\alpha K$

(v) $w(T, x; \alpha) = [\alpha K - (1+\alpha)x]^+ I\!I_{\{x < B\}}$

(vi) $w(t, x; \alpha)$ is continuous on $[0, T] \times [0, B]$.

Now we define in the familiar way

$$v(t,x;\alpha) \stackrel{\Delta}{=} \int_0^1 y^{\alpha-1} w(t,xy;\alpha)dy \tag{139}$$

and derive a list of properties of $v(t,x;\alpha)$:

(i) $v(t,x;\alpha)$ satisfies the Black-Scholes partial differential equation.

(ii) $0 \leq v(t,x;\alpha) \leq K$ and $v(t,0;\alpha) = e^{-r(T-t)}K$.

(iii) $xv_x(t,x;\alpha) + \alpha v(t,x;\alpha) = w(t,x;\alpha)$ and therefore in particular

(iv) $Bv_x(t,B;\alpha) + \alpha v(t,B;\alpha) = 0$.

(v) $v(T,x;\alpha) = \hat{\phi}(x)$

(vi) $v(t,B;\alpha) = \int_0^1 y^{\alpha-1} w(t,By;\alpha)dy$

(vii) $v(t,x;\alpha)$ is continuous on $[0,T] \times [0,B]$.

(viii) $\lim_{x\to 0} xv_x(t,x) = 0$

An explicit analytical solution can be found by first computing

$$w(t,x;\alpha) = (1+\alpha)V(t,x),$$

where V is the value function of an *unconstrained* up-and-out put option with barrier B and strike $K' \stackrel{\Delta}{=} \frac{\alpha}{1+\alpha}K$, and then evaluate equation 139. For the practitioner we derive the formula for $V(t,x)$: We set up the Brownian motion with drift $\tilde{W}(t) \stackrel{\Delta}{=} W(t) + \theta_- t$ and its running minimum $\tilde{m}(t) \stackrel{\Delta}{=} \min_{0\leq u\leq t} \tilde{W}(u)$. Next we look up the joint density of the random pair $(\tilde{W}(T), \tilde{m}(T))$ in [BORODIN and SALMINEN 2.1, formula 1.2.8]:

$$f(\tilde{m},\tilde{w}) = \exp(\theta_- \tilde{w} - \frac{1}{2}\theta_-^2 T)\frac{2(\tilde{w}-2\tilde{m})}{T\sqrt{2\pi T}} \exp\left(-\frac{(\tilde{w}-2\tilde{m})^2}{2T}\right), \tag{140}$$
$$\tilde{m} < 0, \quad \tilde{w} > \tilde{m}, \quad \theta_\pm \stackrel{\Delta}{=} \frac{r}{\sigma} \pm \frac{\sigma}{2}.$$

write

$$\begin{aligned} V(0,S_0) &= I\!\!E\left[e^{-rT}(K'-S_T)^+ I\!\!I_{\{\min_{0\leq t\leq T} S_t \geq B\}}\right] \\ &= e^{-rT} I\!\!E\left[(K'-S_0 e^{\sigma \tilde{W}_T})^+ I\!\!I_{\{S_0 e^{\sigma \min_{0\leq t\leq T} \tilde{W}_t} \geq B\}}\right] \\ &= e^{-rT} \int_{x=\frac{1}{\sigma}\log\frac{B}{x}}^{x=\frac{1}{\sigma}\log\frac{K'}{x}} \int_{y=\frac{1}{\sigma}\log\frac{B}{x}}^{y=x\wedge 0} (K'-S_0 e^{\sigma x})\, f(y,x)\, dy\, dx \end{aligned} \tag{141}$$

The rest is elementary calculus (and patience). We also refer the reader to [RICH] for more formulas and comparative statics of barrier options.

We would now like to prove the identities

$$\begin{aligned} v(0,S_0;\alpha) &= I\!\!E\left[e^{-rT-\alpha\lambda^*(T)}\left(K - S(T)e^{-\lambda^*(T)}\right)^+\right] \\ &= \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} I\!\!E\left[e^{-rT-\alpha\lambda(T)}\left((K-S(T)e^{-\lambda(T)})^+ I\!\!I_{\{S(t)e^{-\lambda(t)}\leq B \forall t\in[0,T]\}}\right)\right] \end{aligned}$$

directly: The second one is obvious, because the minmal effort to make the indicator equal one is reflection at the barrier, which λ^* does. Using this λ^*, the indicator can be dropped. The maximizing $\lambda^*(T)$ is the one we derived in the vanilla put option. Let us now approach the first equality. For a fixed $\lambda \in \mathcal{L}_+^{\mathrm{rc}}$ define the controlled stock price process S^λ, its running supremum M^λ and the first crossing time τ by

$$S^\lambda(0) \triangleq S(0) \tag{142}$$

$$dS^\lambda(t) \triangleq S^\lambda(t)[rdt + \sigma dW(t) - d\lambda(t)] \tag{143}$$

$$M^\lambda(t) \triangleq \sup_{u \le t} S^\lambda(u) \tag{144}$$

$$\tau \triangleq T \wedge \inf\{t \ge 0 : S^\lambda(t) > B\} \tag{145}$$

Furthermore we need to include the dependence of v of the running supremum. So let us define a function $v(t,x,y;\alpha)$ by

$$\begin{aligned} -rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} &= 0, \quad 0 \le t < T, \quad 0 < x < B \\ v_y &= 0, \quad 0 \le t < T, \quad 0 < x \le y \le B \\ \alpha v(t,B,B;\alpha) + Bv_x(t,B,B;\alpha) &= 0, \quad 0 \le t \le T \\ \hat{\phi}(x) \triangleq v(T,x,y;\alpha) &= \begin{cases} K - x & \text{if } x \le B \wedge \frac{\alpha}{1+\alpha}K \\ \frac{K}{1+\alpha}\left(\frac{\alpha K}{(1+\alpha)x}\right)^\alpha & \text{if } B \ge x \ge \frac{\alpha}{1+\alpha}K \\ 0 & \text{if } x > B \end{cases} \\ v &= 0, \quad x > B \end{aligned}$$

We understand that $v_y(t,x,B;\alpha)$ means $\lim_{y\uparrow B} v_y(t,x,y;\alpha)$. Of course, v jumps down to zero, as y crosses the barrier. Now we compute the differential

$$\begin{aligned} & d\left(e^{-rt-\alpha\lambda(t)} v(t, S_t^\lambda, M_t^\lambda;\alpha)\right) \\ = \; & e^{-rt-\alpha\lambda(t)} \left\{\mathcal{L}vdt - (\alpha v + S_t^\lambda v_x)d\lambda(t) + v_y dM_t^\lambda + \sigma S_t^\lambda v_x dW_t\right\} \end{aligned} \tag{146}$$

We integrate from 0 to T and take expecations:

$$\begin{aligned} & \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} v(T, S_T^\lambda, M_T^\lambda;\alpha)\right] - v(0,S_0,S_0;\alpha) \\ = \; & \mathbb{E}\int_0^\tau e^{-rt-\alpha\lambda(t)} \left\{\mathcal{L}vdt - (\alpha v + S_t^\lambda v_x)d\lambda(t) + v_y dM_t^\lambda + \sigma S_t^\lambda v_x dW_t\right\} \\ + \; & \mathbb{E}\int_\tau^T e^{-rt-\alpha\lambda(t)} \left\{\mathcal{L}vdt - (\alpha v + S_t^\lambda v_x)d\lambda(t) + v_y dM_t^\lambda + \sigma S_t^\lambda v_x dW_t\right\} \\ = \; & -\mathbb{E}\int_0^\tau e^{-rt-\alpha\lambda(t)} (\alpha v + S_t^\lambda v_x)d\lambda(t) \\ \le \; & 0 \end{aligned} \tag{147}$$

We conclude that for all $\lambda \in \mathcal{L}_+^{\mathrm{rc}}$

$$v(0,S_0,S_0;\alpha) \ge \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} v(T, S_T^\lambda, M_T^\lambda;\alpha)\right] \tag{148}$$

and hence

$$v(0,S_0,S_0;\alpha) \ge \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} v(T, S_T^\lambda, M_T^\lambda;\alpha)\right]. \tag{149}$$

For $\lambda = \lambda^* \in \mathcal{L}_+^{\text{rc}}$ we obtain

$$\begin{aligned} v(0, S_0, S_0; \alpha) &= \mathbb{E}\left[e^{-rT-\alpha\lambda^*(T)} v(T, S_T^{\lambda^*}, M_T^{\lambda^*}; \alpha)\right] \\ &= \mathbb{E}\left[e^{-rT-\alpha\lambda^*(T)} \left(K - S(T)e^{-\lambda^*(T)}\right)^+\right] \end{aligned} \quad (150)$$

and hence also

$$v(0, S_0, S_0; \alpha) = \sup_{\lambda \in \mathcal{L}_+^{\text{rc}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} v(T, S_T^{\lambda}, M_T^{\lambda}; \alpha)\right]. \quad (151)$$

5.10 Basket of Barrier Options

Let's look at the example of two up and out call options which have different barriers but are identical otherwise:

$$F = (S(T) - K)^+ \mathbb{I}_{\{S(t) \le B_1 \forall t \in [0,T]\}} + (S(T) - K)^+ \mathbb{I}_{\{S(t) \le B_2 \forall t \in [0,T]\}}$$

for some positive strike K and positive barriers $B_2 > B_1 > K$.

$$F^* = (S(T)e^{-\lambda(T)} - K)^+ \mathbb{I}_{\{S(t)e^{-\lambda(t)} \le B_1 \forall t \in [0,T]\}}$$
$$+(S(T)e^{-\lambda(T)} - K)^+ \mathbb{I}_{\{S(t)e^{-\lambda(t)} \le B_2 \forall t \in [0,T]\}}$$

The control problem is

$$u^*(S_0) = \sup_{\lambda \in \mathcal{L}_+^{\text{rc}}} \mathbb{E}\left[e^{-rT-\alpha\lambda(T)} \left((S(T)e^{-\lambda(T)} - K)^+ \mathbb{I}_{\{S(t)e^{-\lambda(t)} \le B_1 \forall t \in [0,T]\}}\right.\right.$$
$$\left.\left.+(S(T)e^{-\lambda(T)} - K)^+ \mathbb{I}_{\{S(t)e^{-\lambda(t)} \le B_2 \forall t \in [0,T]\}}\right)\right].$$

Again, we solve this problem by first characterizing a minimal super-replicating value-function $v(t, x, y; \alpha)$ via a partial differential equation approach and then identify

$$v(0, S_0, S_0; \alpha) = u^*(S_0).$$

As in the example of the up-and-out put, the x-variable stands for the stock price at time t and the y-variable stands for the running supremum up to time t. Now we define $v(t, x, y; \alpha)$ for $t \in [0, T]$, $x \le y \in [0, \infty)$:

(1) for $y > B_2$ both the options have knocked out, so let $v = 0$.

(2) for $y \in (B_1, B_2]$ only the upper barrier option has knocked out, so we let

$$v(t, x, y; \alpha) \triangleq v(t, x; \alpha), \quad (152)$$

where $v(t, x; \alpha)$ is the minimal super-replicating value function of a single up-and-out call option with strike K and barrier B_2 as discussed in our introductory example.

(3) for $y \in [0, B_1]$ define v via

(3.1) $v(T, x, y; \alpha) = 2(x - K)^+$

(3.2) $\mathcal{L}v = 0$ for $(t, x, y) \in (0, T) \times (0, B_1) \times (0, B_1)$.

(3.3) $v(t,0,y;\alpha) = 0$ for $t \in [0,T]$.

(3.4) $\alpha v(t,B_1,B_1;\alpha) + B_1 v_x(t,B_1,B_1;\alpha) = 0$ for $t \in [t^*,T]$, where

$$t^* \triangleq 0 \vee \inf\{t : v(t,B_1,B_1;\alpha) > v(t,B_1;\alpha)\} \in [0,T).$$

On this line segment we will then have $v(t,B_1,B_1;\alpha) \geq v(t,B_1;\alpha)$.

(3.5) $v(t,B_1,B_1;\alpha) = v(t,B_1;\alpha)$ for $t \in [0,t^*]$. On this line segment we will have $\alpha v(t,B_1,B_1;\alpha)+B_1 v_x(t,B_1,B_1;\alpha) \geq 0$, because $v(t,B_1;\alpha)$ has that property.

Observe that $v_y = 0$ except for the downward jumps at the two barriers. It is obvious that $v(t,x,y;\alpha)$ superreplicates the payoff F. Since $\mathcal{L}\{\alpha v(t,x,y;\alpha) + x v_x(t,x,y;\alpha)\} = 0$ and $\alpha v(t,x,y;\alpha) + x v_x(t,x,y;\alpha) \geq 0$ at all boundaries, it follows from the maximum-principle that $v(t,x,y;\alpha)$ satisfies the shortselling constraint everywhere. We need to check that our $v(t,x,y;\alpha)$ is the cheapest such super-replication. We have not increased the payoff at $t = T$ or at $x = 0$. The minimality for $y > B_1$ has been already established in the introductory example. To guarantee super-replication and limited shortselling for $y \in [0,B_1]$, we must have for all $t \in [0,T]$ at B_1:

$$v(t,B_1,B_1;\alpha) \geq v(t,B_1;\alpha) \quad (153)$$

$$\alpha v(t,B_1,B_1;\alpha) + B_1 v_x(t,B_1,B_1;\alpha) \geq 0 \quad (154)$$

For t above t^*, 153 is true and 154 is tight. For this line segment the minimality follows from an argument like in the up-and-out call. For t below t^*, 154 is true and 153 is tight, which means that no lifting has taken place, whence it is certainly minimal. This shows that $v(t,x,y;\alpha)$ solves the constraint valuation problem. It can be evaluated analyctically except for possibly the box $[0,t^*]\times(0,B_1)$. Inside the box it can certainly be solved numerically, because all boundary and terminal conditions are explicitely available. To identify $v(0,S_0,S_0;\alpha)$ with $u^*(S_0)$ we define again for a fixed $\lambda \in \mathcal{L}_+^{\mathrm{rc}}$ the controlled stock price process S^λ, its running supremum M^λ and the first crossing times $\tau_1 < \tau_2$ by

$$S^\lambda(0) \triangleq S(0) \quad (155)$$

$$dS^\lambda(t) \triangleq S^\lambda(t)[rdt + \sigma dW(t) - d\lambda(t)] \quad (156)$$

$$M^\lambda(t) \triangleq \sup_{u \leq t} S^\lambda(u) \quad (157)$$

$$\tau_i \triangleq T \wedge \inf\{t \geq 0 : S^\lambda(t) > B_i\} \quad (158)$$

Notice that $S^\lambda(t) = S(t)e^{-\lambda(t)}$. We define the optimal λ^* by

$$\lambda^*(t) = \max_{u\in[0,t]} [\log S(u) - \log B(u)]^+, \quad (159)$$

$$B(u) = \left\{ \begin{array}{ll} B_2 & \text{if } u < t^* \\ B_2 & \text{if } u \geq t^* \text{and } M_{t^*}^{\lambda^*} > B_1 \\ B_1 & \text{if } u \geq t^* \text{and } M_{t^*}^{\lambda^*} \leq B_1 \end{array} \right\}.$$

Now we compute the differential

$$\begin{aligned} & d\left(e^{-rt-\alpha\lambda(t)} v(t,S_t^\lambda,M_t^\lambda;\alpha)\right) \quad (160) \\ = \; & e^{-rt-\alpha\lambda(t)} \left\{\mathcal{L}v dt - (\alpha v + S_t^\lambda v_x) d\lambda(t) + v_y dM_t^\lambda + \sigma S_t^\lambda v_x dW_t\right\}, \end{aligned}$$

integrate from 0 to T and take expecations:

$$
\begin{aligned}
& \mathbb{E}\left[e^{-rT-\alpha\lambda(T)}v(T,S_T^\lambda,M_T^\lambda;\alpha)\right]-v(0,S_0,S_0;\alpha)\\
= & \ \mathbb{E}\int_0^{\tau_1} e^{-rt-\alpha\lambda(t)}\left\{\mathcal{L}v dt-(\alpha v+S_t^\lambda v_x)d\lambda(t)+v_y dM_t^\lambda+\sigma S_t^\lambda v_x dW_t\right\}\\
+ & \ \mathbb{E}\int_{\tau_1}^{\tau_2} e^{-rt-\alpha\lambda(t)}\left\{\mathcal{L}v dt-(\alpha v+S_t^\lambda v_x)d\lambda(t)+v_y dM_t^\lambda+\sigma S_t^\lambda v_x dW_t\right\}\\
+ & \ \mathbb{E}\int_{\tau_2}^{T} e^{-rt-\alpha\lambda(t)}\left\{\mathcal{L}v dt-(\alpha v+S_t^\lambda v_x)d\lambda(t)+v_y dM_t^\lambda+\sigma S_t^\lambda v_x dW_t\right\}\\
= & -\mathbb{E}\int_0^{\tau_1} e^{-rt-\alpha\lambda(t)}(\alpha v+S_t^\lambda v_x)d\lambda(t)\\
& -\mathbb{E}\int_{\tau_1}^{\tau_2} e^{-rt-\alpha\lambda(t)}(\alpha v+S_t^\lambda v_x)d\lambda(t)\\
\leq & \ 0
\end{aligned}
\tag{161}
$$

We conclude that for all $\lambda\in\mathcal{L}_+^{\mathrm{rc}}$

$$
\begin{aligned}
v(0,S_0,S_0;\alpha) \ \geq & \ \mathbb{E}\left[e^{-rT-\alpha\lambda(T)}v(T,S_T^\lambda,M_T^\lambda;\alpha)\right]\\
= & \ \mathbb{E}\Big[e^{-rT-\alpha\lambda(T)}\left(S(T)e^{-\lambda(T)}-K\right)^+\Big\{\mathbb{I}_{\{S(t)e^{-\lambda(t)}\leq B_1\forall\ t\}}\\
& \quad + \mathbb{I}_{\{S(t)e^{-\lambda(t)}\leq B_B\forall\ t\}}\Big\}\Big]
\end{aligned}
\tag{162}
$$

and hence

$$
v(0,S_0,S_0;\alpha)\geq\sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}}\mathbb{E}\left[e^{-rT-\alpha\lambda(T)}v(T,S_T^\lambda,M_T^\lambda;\alpha)\right]. \tag{163}
$$

For $\lambda=\lambda^*\in\mathcal{L}_+^{\mathrm{rc}}$ we obtain

$$
\begin{aligned}
v(0,S_0,S_0;\alpha) = & \ \mathbb{E}\left[e^{-rT-\alpha\lambda^*(T)}v(T,S_T^{\lambda^*},M_T^{\lambda^*};\alpha)\right] \qquad (164)\\
= & \ \mathbb{E}\left[e^{-rT-\alpha\lambda^*(T)}\left(S(T)e^{-\lambda^*(T)}-K\right)^+\left\{\mathbb{I}_{\{M_T^{\lambda^*}\leq B_1\}}+1\right\}\right]
\end{aligned}
$$

and hence also

$$
\begin{aligned}
v(0,S_0,S_0;\alpha) = & \sup_{\lambda\in\mathcal{L}_+^{\mathrm{rc}}}\mathbb{E}\left[e^{-rT-\alpha\lambda(T)}v(T,S_T^\lambda,M_T^\lambda;\alpha)\right]\\
= & \ u^*(S_0).
\end{aligned}
\tag{165}
$$

Here is some numerical information: We compute some values of t^* for the basic choice of parameters $K=1.40$, $B_1=1.50$, $B_2=1.52$, $\sigma=8\%$, $r_d=5\%$, $r_f=0\%$. Then we get

1. $t*=12$ days
2. $t*=7$ days when we change the strike to $K=1.45$
3. $t*=10$ days when we change the upper barrier to $B_2=1.55$

4. $t* = 14$ days when we change the upper barrier to $B_2 = 1.51$
5. $t* = 46$ days when we change the volatility to $\sigma = 3\%$
6. $t* = 2$ days when we change the volatility to $\sigma = 20\%$
7. $t* = 13$ days when we change the domestic interest rate to $r_d = 0\%$

This list of examples can be carried on arbitrarily as there can be options invented arbitrarily. We would like to point out though, that a solution is certainly not always so easy to find. Already, when we think of Down-and-In Puts or Down-and-In Calls, we run into free boundary problems to get the cheapest super-replicating value-function v. Our thesis provides at least a way to check the answer. We leave their solution to the curiosity of future researchers.

6 Appendix

6.1 Properties of Weak Convergence

A sequence of functions $\{\lambda_n\}$ in Λ_+^{rc} is said to converge weakly to a function $\lambda \in \Lambda_+^{\mathrm{rc}}$, if $\lim_n \lambda_n(t) = \lambda(t)$ for each continuity point t of λ *and* for $t \in \{0, T\}$. Convergence at zero is redundant, because all $\lambda \in \Lambda_+^{\mathrm{rc}}$ start at zero. We will now list some properties and implications:

Theorem 6.1 *Weak convergence is equivalent to the requirement that for each $\phi \in \mathbf{C}[0,T]$:*

$$\lim_{n\to\infty} \int_0^T \phi(t)\, d\lambda_n(t) = \int_0^T \phi(t)\, d\lambda(t).$$

Proof. Viewing Λ_+^{rc} as a space of finite measures on $[0, T]$, this is part of the *Portmanteaux theorem* (II.6.1 of Parthasarathy).

Theorem 6.2 *Each $\lambda \in \Lambda_+^{\mathrm{rc}}$ has only finitely many jumps of a given size and hence at most countably many jumps.*

Proof. Lemma VII.6.2 and the remark after that of Parthasarathy.

Theorem 6.3 *For $a \in [0, T]$ we have*

$$\lambda(a-) \le \liminf \lambda_n(a) \le \limsup \lambda_n(a) \le \lambda(a) = \lambda(a+).$$

Proof. The second inequality is trivial. The first is analogous to the third. To prove the third, first note that due to convergence at the endpoints, we only need to consider $a \in (0, T)$. Assume $B \triangleq \limsup \lambda_n(a) > \lambda(a) \triangleq A$ and set $C \triangleq \frac{A+B}{2}$. Since λ is right-continuous, there exists a number $c \in (a, T)$, such that $\lambda(t) < C$ for all $t \in [a, c]$. $[a, c]$ must contain a continuity point of λ, say b. It follows that $\lambda_n(a) \le \lambda_n(b)$ for all n and consequently

$$\limsup \lambda_n(a) \le \limsup \lambda_n(b) = \lambda(b) < C < \limsup \lambda_n(a),$$

which is a contradiction.

Theorem 6.4 *For a sequence $\{a_n\}_n \in [0,T]$ converging to a we have*

$$\lambda(a-) \leq \liminf \lambda_n(a_n) \leq \limsup \lambda_n(a_n) \leq \lambda(a).$$

Proof. Again it is sufficient to prove only the last inequality.

Case I: $a_n \leq a$ except for finitely many n. Then there exists an integer N such that $a_n \leq a$ for all $n \geq N$. Therefore, $\lambda_n(a_n) \leq \lambda_n(a)$ for all $n \geq N$ and thus by theorem 6.3 $\limsup \lambda_n(a_n) \leq \limsup \lambda_n(a) \leq \lambda(a)$.

Case II: There are infinitely many n, such that $a_n > a$. These form a subsequence $\{a'_k\}$, converging to a from the right. Fix k. For a'_k there exists an integer $N(k)$, such that $a_n \leq a'_k$ for all $n \geq N(k)$. This implies $\lambda_n(a_n) \leq \lambda_n(a'_k)$ for all $n \geq N(k)$ and thus $\limsup \lambda_n(a_n) \leq \limsup \lambda_n(a'_k) \leq \lambda(a'_k)$. This is true for all k, so taking the limit we get

$$\limsup \lambda_n(a_n) \leq \lim_k \lambda(a'_k) \leq \lambda(a+) = \lambda(a).$$

Theorem 6.5 *Given a sequence $\{\lambda_n\}_n \in \Lambda_+^{\text{rc}}$, for which $\sup_n \lambda_n(T) < \infty$, there exists a $\lambda \in \Lambda_+^{\text{rc}}$ and a subsequence $\{\lambda_{n_k}\}_k \in \Lambda_+^{\text{rc}}$, which converges weakly to λ.*

Proof. This is really a statement about relative compactness of a tight sequence of probability measures.

Case I: $\lambda_n(T) = \lambda_n(0)$ for infinitely many n. Then for these n we already have a subsequence converging weakly to $\lambda(t) \triangleq \lambda(0)$.

Case II: $\lambda_n(T) = \lambda_n(0)$ for only finitely many n. Since the sequence $\{\lambda_n(T)\}_n$ is bounded, it must have an accumulation point $\lambda(T)$ and a subsequence converging to $\lambda(T)$. For notational convenience, we name this subsequence again $\{\lambda_n(T)\}_n$, and assume without loss of generality that $\lambda_n(T) > \lambda_n(0)$ for all n. Now we normalize to probability measures: Define

$$\tilde{\lambda}_n(t) \triangleq \frac{\lambda_n(t) - \lambda_n(0)}{\lambda_n(T) - \lambda_n(0)}.$$

Observe that $\tilde{\lambda}_n(t) \in \Lambda_+^{\text{rc}}$ and additionally $\tilde{\lambda}_n(0) = 0$ and $\tilde{\lambda}_n(T) = 1$. By theorem 6.1. of Billingsley, a subsequence $\{\tilde{\lambda}_{n_k}\}_k$ converges weakly to a $\tilde{\lambda} \in \Lambda_+^{\text{rc}}$. To go back, define

$$\lambda(t) \triangleq [\lambda(T) - \lambda(0)]\tilde{\lambda}(t) + \lambda(0) \in \Lambda_+^{\text{rc}}$$

and notice that λ and $\tilde{\lambda}$ have the same points of continuity. Finally, for any continuity point t of λ,

$$\lambda_{n_k}(t) = [\lambda_{n_k}(T) - \lambda_{n_k}(0)]\tilde{\lambda}_{n_k}(t) + \lambda_{n_k}(0)$$

converges to $\lambda(t)$.

Remark: If $\{\lambda_n\}_n$ is a sequence of processes, then one can do this for every ω; however, we get different subsequences for different ω.

Theorem 6.6 Λ_+^{c} *is dense in* Λ_+^{rc} *under the weak topology.*

Proof. We may do mollifications to the right. Let $\lambda \in \Lambda_+^{\mathrm{rc}}$ be given. We want to approximate it by a sequence $\{f_n\}_n$ in Λ_+^{c}. The idea is: at each $t \in [0,T]$ take $f_n(t)$ to be the weighted average of λ over the interval $[t, t+\frac{1}{n}]$. To do this, first extend λ beyond T, by setting $\lambda(t) \triangleq \lambda(T)$ for $t > T$. Let the weight be any nonnegative probability density function $\varphi \in \mathbf{C}^\infty(I\!R)$, whose support is the interval $[0,1]$ and $\int_0^1 \varphi(t)\,dt = 1$. As an example one can take

$$\varphi(t) = C_\varphi \begin{cases} e^{\frac{1}{(2t-1)^2-1}} & \text{if } t \in [0,1] \\ 0 & \text{if } t \notin [0,1] \end{cases}$$

for an appropriate normalizing constant C_φ. [BREZIS IV.4, p.70] Now define

$$f_n(t) \triangleq \int_0^1 \lambda(t + \frac{y}{n})\varphi(y)dy \;=\; n\int_t^{t+\frac{1}{n}} \lambda(s)\varphi(n(s-t))ds.$$

This shows that f_n is continuous, $\lambda \le f_{n+1} \le f_n$. Furthermore the right-continuity of λ and the bounded convergence theorem guarantee that $f_n(t)$ converges to $\lambda(t)$ for all $t \in [0,T]$. We have created pointwise monotone convergence from above. There can be other ways to do this. If we mollify to the left instead, we get pointwise monotone convergence from below to the leftcontinous version of λ, which is still weak convergence, unless λ jumps at T. We can also do mollifications with moving attention spans in order to keep finitely many points of λ fixed. See below for details.

Theorem 6.7 *The* Lévy distance $d(\mu, \lambda)$ *of two controls* μ *and* λ *is defined to be*

$$d(\mu,\lambda) \triangleq \inf\{\epsilon > 0 : \lambda(t-\epsilon) - \epsilon \le \mu(t) \le \lambda(t+\epsilon) + \epsilon \text{ for all } t\ \}.$$

This is a metric on Λ_+^{rc} *and a necessary and sufficient condition for* λ_n *converging weakly to* λ *is that* $d(\lambda_n, \lambda) \to 0$. *Moreover,* Λ_+^{rc} *is separable in the Lévy metric.*

Proof. See Billingsley, Probability and Measure, 2nd ed. Problems 14.9 and 25.4.

Theorem 6.8 *(*Dini's theorem*) establishes the following relation between pointwise and uniform convergence: Let* $f_n \in \mathbf{C}[0,T]$, $f \in \mathbf{C}[0,T]$, $f_n \uparrow f$ *pointwise or* $f_n \downarrow f$ *pointwise, then the convergence is uniform.*

Proof. See Heuser, Lehrbuch der Analysis, 108.1. The compactness of $[0,T]$ is the essential ingredient.

Theorem 6.9 *(Variation of*Dini's theorem*) Let* $\lambda_n \in \Lambda_+^{\mathrm{rc}}$, $f \in \mathbf{C}[0,T]$, $\lambda_n \to f$ *weakly (i.e. pointwise) , then the convergence is uniform.*

Proof. f must be nondecreasing and uniformly continuous.

Case I: $f(T) = f(0)$. Then $f(t) = f(0)$ for all $t \in [0,T]$. Given $\epsilon > 0$ there exists an integer N, such that for all $n \ge N$ both $|\lambda_n(0) - f(0)| < \epsilon$ and $|\lambda_n(T) - f(0)| < \epsilon$. Since λ_n is nondecreasing, we conclude that $|\lambda_n(t) - f(0)| < \epsilon$ for all t and for all $n \ge N$.

Case II: $f(T) > f(0)$. Define $\delta \triangleq f(T) - f(0)$. Since we always have convergence at the endpoints, there must be an integer N, such that for all $n \geq N$, $\lambda_n(T) - \lambda_n(0) > \frac{\delta}{3} > 0$. To prove convergence it is sufficient to assume that the sequence starts at N. Now we translate all the functions into the world of distribution functions by defining

$$F(t) \triangleq \frac{f(t) - f(0)}{f(T) - f(0)},$$

$$\Lambda_n(t) \triangleq \frac{\lambda_n(t) - \lambda_n(0)}{\lambda_n(T) - \lambda_n(0)}.$$

The pointwise convergence of λ_n to λ implies that $\Lambda_n(t) \to F(t)$ for all t. Hence $d(\Lambda_n, F) \to 0$ as $n \to \infty$, where d is the Lévy distance of the space of distribution functions. This in turn means that for all $\epsilon > 0$ there is an integer N, such that for all $n \geq N$ we have $F(t - \epsilon) - \Lambda_n(t) \leq \epsilon$ and $\Lambda_n(t) - F(t + \epsilon) \leq \epsilon$. Introduce the *modulus of continuity* of a continuous function f as

$$\delta_f(\epsilon) \triangleq \sup_{|x-y|\leq\epsilon} |f(x) - f(y)|.$$

We know that for a uniformly continuous function f, $\delta_f(\epsilon) \to 0$ as $\epsilon \to 0$. Now we can conclude that for all t and all $n \geq N$:

$$F(t) - \Lambda_n(t) = F(t - \epsilon) - \Lambda_n(t) + F(t) - F(t - \epsilon) \leq \epsilon + \delta_f(\epsilon)$$

and

$$\Lambda_n(t) - F(t) = F(t + \epsilon) - F(t) + \Lambda_n(t) - F(t + \epsilon) \leq \delta_f(\epsilon) + \epsilon$$

Together for all $n \geq N$

$$\sup_{t\in[0,T]} |F(t) - \Lambda_n(t)| \leq \delta_f(\epsilon) + \epsilon.$$

The right hand side goes to zero as $\epsilon \downarrow 0$. This proves $\Lambda_n \to F$ uniformly and yields $1 - \Lambda_n \to 1 - F$ uniformly. Since by definition

$$f(t) = f(T)F(t) + f(0)(1 - F(t))$$

and

$$\lambda_n(t) = \lambda_n(T)\Lambda_n(t) + \lambda_n(0)(1 - \Lambda_n(t))$$

and $\{\lambda_n(0)\}_n$ and $\{\lambda_n(T)\}_n$ are sequences of real numbers converging to $\lambda(0)$ and $\lambda(T)$ respectively, we conclude the uniform convergence of $\lambda_n \to \lambda$.

Remark. The monotonicity of the functions λ_n cannot be waived, since for example the sequence of even smooth functions $f_n(t) \triangleq nt(1-t)^n$ converges pointwise to the smooth function $f(t) = 0$, but each f_n has a maximum at the point $(\frac{1}{n+1}, (\frac{n}{n+1})^{n+1})$, whose y-coordinate converges to $\frac{1}{e} > 0$ as $n \to \infty$.

Theorem 6.10 *(*Mollification with moving attention spans*) We are given a $\lambda \in \Lambda_+^{\mathrm{rc}}$ and a subset $\{t_1, t_2, \dots t_N\}$ of $[0,T]$, such that $0 \le t_1 < t_2 < \dots < t_N \le T$. This can be weakly approximated by a sequence $\{f_n\}_n \in \Lambda_+^{\mathrm{c}}$ preserving the values at all the t_n.*

Proof. For each $k = 1, 2, \dots, N-1$ define $\lambda_k \in \Lambda_+^{\mathrm{rc}}$ by

$\lambda_k(t) \triangleq \lambda(t_k)$ for $t \le t_k$.

$\lambda_k(t) \triangleq \lambda(t)$ for $t_k \le t \le t_{k+1}$.

$\lambda_k(t) \triangleq \lambda(t_{k+1})$ for $t \ge t_{k+1}$.

Also define

$$l_{k,n}(t) \triangleq \frac{t - t_{k+1}}{t_k - t_{k+1}}(t_k - t_{k+1} - \frac{1}{n}) + t_{k+1}.$$

This implies

$l'_{k,n}(t) = 1 + \frac{1}{n(t_{k+1}-t_k)} > 1$, so $l_{k,n}(t)$ increases faster than $id(t) = t$.

$l_{k,n}(t_k) = t_k - \frac{1}{n}$.

$l_{k,n}(t_{k+1}) = t_{k+1}$.

Using the weight φ of theorem 6.6 the desired sequence can be defined as

$$f_n(t) \triangleq \int_0^1 \lambda_k(l_{k,n}(t) + \frac{y}{n})\, \varphi(y)\, dy \quad \text{if } t \in [t_k, t_{k+1}].$$

Interpretation: Each attention span has length $\frac{1}{n}$ and starting point $l_{k,n}(t)$. At the beginning of the interval $[t_k, t_{k+1}]$ we do mollification to the left, at the end of it we do mollification to the right. One can instantly verify the following list of properties of the sequence $\{f_n\}_n$:

$f_n \in \mathbf{C^1}[0,T]$.

$\lim f_n(t) = \lambda(t)$, if λ is continuous at t.

f_n is nondecreasing.

$f_n(t_k) = \lambda(t_k)$ for all $k = 1, 2, \dots, N$.

Theorem 6.11 *We are given a sequence $\{f_n\}_n \in \Lambda_+^{\mathrm{c}}$ converging weakly to $\lambda \in \Lambda_+^{\mathrm{rc}}$. We are also given a $g \in \mathbf{C}[0,T]$. We define*

$M \triangleq \sup_{t\in[0,T]}(g(t) - \lambda(t)), \;\; M_n \triangleq \max_{t\in[0,T]}(g(t) - f_n(t))$

$m \triangleq \inf_{t\in[0,T]}(g(t) - \lambda(t)), \;\; m_n \triangleq \min_{t\in[0,T]}(g(t) - f_n(t))$

$A \triangleq \int_0^T \exp[g(t) - \lambda(t)]\, dt, \;\; A_n \triangleq \int_0^T \exp[g(t) - f_n(t)]\, dt$

It follows that $\lim M_n = M$, $\lim m_n = m$ and $\lim A_n = A$.

Proof. To prove $\lim M_n = M$, let us first observe that for all $s \in [0, T]$ and some $t \in [0, T]$

$$M_n = \max_{t \in [0,T]} (g(t) - f_n(t)) = g(t) - f_n(t) \geq g(s) - f_n(s)$$

and that similarly for all $s \in [0, T]$ and some $t \in [0, T]$

$$M = \sup_{t \in [0,T]} (g(t) - \lambda(t)) = g(t) - \lambda(t-) \geq g(s) - \lambda(s-).$$

(a) $\liminf M_n \geq M$: For all n and all s, we have $M_n \geq g(s) - f_n(s)$. Taking the limes inferior on both sides, we derive

$$\liminf M_n \geq g(s) - \limsup f_n(s) \geq g(s) - \lambda(s),$$

where the last inequality follows from theorem 6.3. We can now take the supremum over all $s \in [0, T]$ on the right hand side and conclude

$$\liminf M_n \geq \sup_{s \in [0,T]} (g(s) - \lambda(s)) = M.$$

(b) $\limsup M_n \leq M$: By the definition of supremum, there is for each n, $\epsilon > 0$ a number s_n such that $M_n \leq g(s_n) - f_n(s_n) + \epsilon$. We choose a subsequence $\{M_{n_j}\}_j$ of $\{M_n\}_n$ such that $\limsup_n M_n = \lim_j M_{n_j}$. The corresponding sequence $\{s_{n_j}\}_j$ is bounded and must therefore have a subsequence $\{s_{n_{j_k}}\}_k$ which converges to a number $s \in [0, T]$. Now

$$\limsup M_n = \lim_j M_{n_j} = \limsup_k M_{n_{j_k}} \geq g(s) - \liminf_k f_{n_{j_k}}(s_{n_{j_k}}) + \epsilon$$

$$\leq g(s) - \lambda(s-) + \epsilon \leq M + \epsilon,$$

where the second last inequality follows from theorem 6.4. This works, because any subsequence of $\{f_n\}_n$ also converges weakly to λ. Now let $\epsilon \downarrow 0$.

To prove $\lim m_n = m$, let us first observe that for all $s \in [0, T]$ and some $t \in [0, T]$

$$m_n = \min_{t \in [0,T]} (g(t) - f_n(t)) = g(t) - f_n(t) \leq g(s) - f_n(s)$$

and that similarly for all $s \in [0, T]$ and some $t \in [0, T]$

$$m = \inf_{t \in [0,T]} (g(t) - \lambda(t)) = g(t) - \lambda(t) \leq g(s) - \lambda(s).$$

(a) $\limsup m_n \leq m$: For all n and all s, we have $m_n \leq g(s) - f_n(s)$. Taking the limes superior on both sides, we derive

$$\limsup m_n \leq g(s) - \liminf f_n(s) \leq g(s) - \lambda(s-),$$

where the last inequality follows from theorem 6.3. We can now take the infimum over all $s \in [0, T]$ on the right hand side and conclude

$$\limsup m_n \leq \inf_{s \in [0,T]} (g(s) - \lambda(s-)) = \inf_{s \in [0,T]} (g(s) - \lambda(s)) = m.$$

(b) $\liminf m_n \geq m$: By the definition of infimum, there is for each n, $\epsilon > 0$ a number s_n such that $m_n \geq g(s_n) - f_n(s_n) - \epsilon$. We choose a subsequence $\{m_{n_j}\}_j$ of $\{m_n\}_n$ such that $\liminf_n m_n = \lim_j m_{n_j}$. The corresponding sequence $\{s_{n_j}\}_j$ is bounded and must therefore have a subsequence $\{s_{n_{j_k}}\}_k$ which converges to a number $s \in [0, T]$. Now

$$\liminf m_n = \lim_j m_{n_j} = \liminf_k m_{n_{j_k}} \geq g(s) - \limsup_k f_{n_{j_k}}(s_{n_{j_k}}) - \epsilon$$

$$\geq g(s) - \lambda(s) - \epsilon \geq m - \epsilon,$$

where the second last inequality follows from theorem 6.4. This works, because any subsequence of $\{f_n\}_n$ also converges weakly to λ. Now let $\epsilon \downarrow 0$.

To prove $\lim A_n = A$, first note that $f_n(t)$ converges to $\lambda(t)$ for Lebesgue-a.e. t. Consequently

$$\int_0^T \lim_{n\to\infty} \exp[g(t) - f_n(t)]\, dt = \int_0^T \exp[g(t) - \lambda(t)]\, dt.$$

Fatou's lemma implies

$$\liminf_{n\to\infty} A_n \geq A.$$

Now select a subsequence $\{A_{n_k}\}_k$ of $\{A_n\}_n$, such that $\limsup_n A_n = \lim_k A_{n_k}$. Since $\exp[g(t) - f_n(t)] \leq \exp[g(t)]$, which is an integrable function, the Dominted Convergence Theorem yields

$$\begin{aligned}
\limsup_{n\to\infty} A_n &= \lim_{k\to\infty} \int_0^T \exp[g(t) - f_{n_k}(t)]\, dt \\
&= \int_0^T \lim_{k\to\infty} \exp[g(t) - f_{n_k}(t)]\, dt \\
&= \int_0^T \exp[g(t) - \lambda(t)]\, dt.
\end{aligned}$$

6.2 The Maximum Principle

The Maximum Principle (see e.g. KARATZAS and SHREVE, 1989) works like this: Suppose X is a diffusion of the form $dX_s = a ds + \sigma dW_s$ with second order differential operator $\mathcal{A}u(t,x) \triangleq a u_x(t,x) + \frac{1}{2}\sigma^2 u_{xx}(t,x)$, $g(t,x) \geq 0$ a *potential*, $u(t,x)$ a function satisfying $u(T,x) \geq 0$ and $-u_t + ru = \mathcal{A}u + g$. Then $u(t,x) \geq 0$ for all $t \leq T$.
For a quick proof, use Itô's rule to compute the differential

$$\begin{aligned}
de^{-rs}u(s, X(s)) &= e^{-rs}[-ruds + u_s ds + \mathcal{A}uds + \sigma u_x dW_s] \\
&= e^{-rs}[-gds + \sigma u_x dW_s]
\end{aligned}$$

Now integrate between t and T and take expectations conditioned on $X(t) = x$ to get

$$I\!E^{t,x}[e^{-rT}u(T, X(T))] = e^{-rt}u(t,x) + I\!E^{t,x}\int_t^T e^{-rs}(-g(s, X(s)))ds,$$

which in turn implies

$$u(t,x) = e^{rt}\left\{ \mathbb{E}^{t,x}[e^{-rT}u(T,X(T))] + \mathbb{E}^{t,x}\int_t^T e^{-rs}g(s,X(s))ds \right\}.$$

The assumed nonnegativity of both $u(T,x)$ and g shows thus, that $u(t,x)$ is nonnegative as well. Throuout we will use this for the case $g = 0$.

6.3 Upper and Lower Semicontinuity

Let F be a real-valued function defined on some metric space (C,d). F is defined to be upper semicontinuous, if

$$\limsup_{n\to\infty} F(x_n) \leq F(x), \text{ whenever } \lim_{n\to\infty} d(x_n,x) = 0$$

and lower semicontinuous, if

$$\liminf_{n\to\infty} F(x_n) \geq F(x), \text{ whenever } \lim_{n\to\infty} d(x_n,x) = 0.$$

Theorem 6.12 *If F is both lower semicontinuous and upper semicontinuous, then F is continuous.*

Theorem 6.13 *F is lower semicontinuous if and only if for each real number a the set*

$$\{x : F(x) \leq a\}$$

is closed or equivalently the set

$$\{x : F(x) > a\}$$

is open.

Theorem 6.14 *F is upper semicontinuous if and only if for each real number a the set*

$$\{x : F(x) \geq a\}$$

is closed or equivalently the set

$$\{x : F(x) < a\}$$

is open.

Theorem 6.15 *F is upper semicontinuous and bounded above if and only if there exists a sequence $\{F_n\}_n$ of continuous functions such that $F_n \downarrow F$.*
F is lower semicontinuous and bounded below if and only if there exists a sequence $\{F_n\}_n$ of continuous functions such that $F_n \uparrow F$.

Proof. See [BERTSEKAS and SHREVE], lemma 7.14.

Warning. If (C_0,d) is a dense metric subspace of (C,d), F agrees with G on C_0 and both F and G are upper semicontinuous, then they need not necessarily agree on C. Take for instance, $C = \mathbb{R}$, $C_0 = \mathbb{Q}$, $y \in \mathbb{R} \setminus \mathbb{Q}$, $F = 0$ and $G(x) = \mathbb{I}_{\{x=y\}}$.

6.4 Excursion on Singular Stochastic Control

Working on Singular Stochastic Control problems may be quite tedious for the beginner in this area. To understand some of the relevant issues we present this easily accessable example. Certainly, in the literature a lot more general problems have been discussed and solved. This excursion is more of an instructional value than to generalize.

The Problem. Let $\{W(t)\}_{t\geq 0}$ be a *Standard Brownian Motion*, $x \in \mathbb{R}$ a *starting point*, α a positive real number. These are given and not to be altered. Let $\{\xi(t)\}_{t\geq 0}$ a nonnegative nondecreasing *control process* starting at zero, adapted to the filtration generated by $W(t)$. These are the variables among which an optimal will have to be chosen. For each *state process*

$$X(t) \triangleq x + W(t) - \xi(t)$$

we are assigned a *cost*

$$C \triangleq \int_0^\infty e^{-\alpha t} X_t^2 \, dt.$$

Define the *minimal expected cost*

$$v(x) \triangleq \min_{\xi} \mathbb{E} \int_0^\infty e^{-\alpha t} X_t^2 \, dt.$$

The problem is

(a) to find a control process ξ^*, for which the above minimum is attained and

(b) to identify the function $v(x)$.

Interestingly, (a) and (b) are somehow interrelated. To interpret the problem, at each time a cost-minimizing controller can push down the Brownian path. Ideally one would like to keep the path at zero, but unfortunately we are not allowed to push up, if $W(t)$ takes large negative values. We have to find the ideal way of pushing down but not pushing down too much.

The Solution. We start off with some elementary observations about $v(x)$:

(1) $v(x) \geq 0$.

(2) $v(x) \leq \frac{x^2}{\alpha} + \frac{1}{\alpha^2}$, because if we don't control at all the expected cost becomes $\int_0^\infty e^{-\alpha t} \mathbb{E}[x + W_t^2] \, dt = \int_0^\infty e^{-\alpha t}[x^2 + t] \, dt = \frac{x^2}{\alpha} + \frac{1}{\alpha^2}$.

(3) $v(x+h) \leq v(x)$ for all $h \geq 0$, because jumping back to x immediately and then optimize is at most optimal.

(4) v is convex: Let ξ and η be the minimizing processes for the starting points x and y respectively, and let $p, q \in [0,1]$ such that $p + q = 1$. At this point we are not concerned about the existence of such optimal control processes. The control process $p\xi + q\eta$ is at most optimal for the starting point $px + qy$. Moreover, the linearity of the state process equation and the convexity of the function $x \mapsto x^2$ yield

$$v(px + qy) \leq \mathbb{E} \int_0^\infty e^{-\alpha t} [px + qy + W_t - (p\xi + q\eta)]^2 \, dt$$

$$= \mathbb{E} \int_0^\infty e^{-\alpha t} [p(x + W_t - \xi) + q(y + W_t - \eta)]^2 \, dt$$

$$\leq p\mathbb{E} \int_0^\infty e^{-\alpha t} (x + W_t - \xi)^2 \, dt + q\mathbb{E} \int_0^\infty e^{-\alpha t} (y + W_t - \eta)^2 \, dt$$

$$= pv(x) + qv(y).$$

(5) Since v is defined and finite for all x, (4) implies that v is continuous.

Now to develop an idea how v and ξ^* should look like, we do a *dynamic programming argument* similar to the one presented in section V.7. of [ROGERS and WILLIAMS]. Getting the right idea is one thing. Proving it is another thing. So we assume for a moment that we can only push down at a *rate* $u_t \in [0, K]$, i.e.

$$dX_t = dW_t + u_t \, dt.$$

This is actually not the case, because we explicitly allow the control processes to have jumps. We now split the controlling over the interval $[0, \infty)$ into two parts: push at a constant rate u over the interval $[0, h]$ and then optimize over $[h, \infty)$. We think of h as a small number.

$$\begin{aligned}
v(x) &= \min_\xi \mathbb{E} \int_0^\infty e^{-\alpha t} (x + W_t - \xi_t)^2 \, dt \\
&\leq \mathbb{E} \left[\int_0^h e^{-\alpha t} (x - ut + W_t)^2 \, dt \right. \\
&\quad \left. + \, e^{-\alpha h} \int_0^\infty e^{-\alpha t} (x - uh + W_h + W_t - \xi_t)^2 \, dt \right] \\
&= \int_0^h e^{-\alpha t} ((x - ut)^2 + t) \, dt + e^{-\alpha h} \mathbb{E} v(x - uh + W_h) \\
&= hx^2 + o(h) + (1 - \alpha h + o(h)) \mathbb{E}[v(x) + v'(x)(-uh + W_h) \\
&\quad + \frac{1}{2} v''(x)(-uh + W_h)^2 + o(h)] \\
&= hx^2 + o(h) + v(x) + h\{-\alpha v(x) - v'(x)u + \frac{1}{2} v''(x)\} + o(h).
\end{aligned}$$

It follows that

$$h(x^2 - \alpha v(x) - v'(x)u + \frac{1}{2} v''(x)) \geq 0.$$

This suggests that either

$$v' = 0 \text{ and } x^2 - \alpha v(x) + \frac{1}{2} v''(x) \geq 0, \; u \text{ active}$$

or

$$v' < 0 \text{ and } x^2 - \alpha v(x) + \frac{1}{2} v''(x) = 0, \; u \text{ inactive}.$$

Recall that $v' > 0$ is not possible. We will now setup a list of desired properties of solution-candidates v and ξ^*, check whether there exists a pair (v, ξ^*) that satisfies all the properties, and then do something that's called a *verification theorem*, where we will check if the solution-candidate actually solves the control

problem.
Determine a function $v(x)$, a number B and a nonnegative nondecreasing process $\xi^*(t)$ that satisfy the

Solution-Candidate Wish-List

(1) $-\alpha v + \frac{1}{2}v'' = -x^2 \ \forall \ x \le B$

(2) $-\alpha v + \frac{1}{2}v'' \ge -x^2 \ \forall \ x > B$

(3) $v' \le 0 \ \forall \ x \le B$

(4) $v' = 0 \ \forall \ x > B$

(5) $0 \le v(x) \le \frac{x^2}{\alpha} + \frac{1}{\alpha^2} \ \forall \ x$

(6) v, v', v'' are continuous for all x.

(7) $\xi^*(t) = 0$, whenever $x + W(t) \le B$.

(8) $\xi^*(t)$ pushes $x + W(t)$ back to B, whenever $x + W(t) > B$.

Find the Solution-Candidate

We first look at the non-homogeneous ordinary differential equation

$$-\alpha v + \frac{1}{2}v'' = -x^2.$$

The corresponding homogeneous ordinary differential equation

$$-\alpha v + \frac{1}{2}v'' = 0$$

has the general solution

$$v(x) = A_1 e^{-\sqrt{2\alpha}x} + A_2 e^{+\sqrt{2\alpha}x}.$$

A particular solution to the non-homogeneous ordinary differential equation can be found be assuming that $v(x) = Ax^2 + Bx + C$. We derive $v'(x) = 2Ax + B$ and $v''(x) = 2A$, whence

$$-x^2 = -\alpha v(x) + \frac{1}{2}v''(x) = -\alpha A x^2 - \alpha B x - \alpha C + A$$

and consequently $A = \frac{1}{\alpha}$, $B = 0$ and $C = \frac{1}{\alpha^2}$. Now we can write down the general solution to the non-homogeneous equation

$$v(x) = A_1 e^{-\sqrt{2\alpha}x} + A_2 e^{+\sqrt{2\alpha}x} + \frac{x^2}{\alpha} + \frac{1}{\alpha^2}.$$

We need this equation to hold for all $x \le B$. But as $x \to -\infty$, (5) can only hold if $A_1 = 0$. To incorporate B, v must be of the form

$$v(x) = \begin{cases} A_2 e^{\sqrt{2\alpha}x} + \frac{x^2}{\alpha} + \frac{1}{\alpha^2} & \text{if } x \le B, \\ v(B) & \text{if } x \ge B. \end{cases}$$

We derive

$$v'(x) = [A_2\sqrt{2\alpha}e^{\sqrt{2\alpha}x} + \frac{2x}{\alpha}]\mathbb{I}_{\{x \le B\}},$$

$$v''(x) = [A_2 2\alpha e^{\sqrt{2\alpha}x} + \frac{2}{\alpha}] \mathbb{I}_{\{x \le B\}}.$$

The continuity of v'' implies that $v''(B) = 0$ and thus $A_2 = -\frac{e^{-\sqrt{2\alpha}B}}{\alpha^2}$ yielding

$$v'(x) = [-\frac{\sqrt{2\alpha}}{\alpha^2} e^{\sqrt{2\alpha}(x-B)} + \frac{2x}{\alpha}] \mathbb{I}_{\{x \le B\}}.$$

The continuity of v' implies that $v'(B) = 0$ and thus $B = \frac{1}{\sqrt{2\alpha}}$. Using the continuity of v we can propose

$$v(x) = \begin{cases} \frac{1}{\alpha^2}[1 + \alpha x^2 - e^{\sqrt{2\alpha}x - 1}] & \text{if } x \le \frac{1}{\sqrt{2\alpha}}, \\ \frac{1}{2\alpha^2} & \text{if } x \ge \frac{1}{\sqrt{2\alpha}}. \end{cases}$$

$$v'(x) = \begin{cases} \frac{1}{\alpha^2}[2\alpha x - \sqrt{2\alpha} e^{\sqrt{2\alpha}x - 1}] & \text{if } x \le \frac{1}{\sqrt{2\alpha}}, \\ 0 & \text{if } x \ge \frac{1}{\sqrt{2\alpha}}. \end{cases}$$

$$v''(x) = \begin{cases} \frac{1}{\alpha^2}[2\alpha - 2\alpha e^{\sqrt{2\alpha}x - 1}] & \text{if } x \le \frac{1}{\sqrt{2\alpha}}, \\ 0 & \text{if } x \ge \frac{1}{\sqrt{2\alpha}}. \end{cases}$$

As expected, we observe that $v''(x) \ge 0$. Therefore v' is nondecreasing. Since $v'(\frac{1}{\sqrt{2\alpha}}) = 0$, $v'(x) \le 0$ for all x. This in turn implies that v must be nonincreasing (as expected). Since $v(\frac{1}{\sqrt{2\alpha}}) = \frac{1}{2\alpha^2}$, we deduce that $v(x) \ge \frac{1}{2\alpha^2}$ for all x. Of course, it is obvious that $v(x) \le \frac{x^2}{\alpha} + \frac{1}{\alpha^2}$, i.e. (5) holds. Finally, if $x \ge \frac{1}{\sqrt{2\alpha}}$, then $x^2 \ge \frac{1}{2\alpha}$ and thus $-\alpha v(x) + \frac{1}{2} v''(x) = -\alpha \frac{1}{2\alpha^2} = -\frac{1}{2\alpha} \ge -x^2$. As a result we can state that properties (1) to (5) of our wish-list are satisfied. Additionally we find $v(0) = \frac{1}{\alpha^2}[1 - \frac{1}{e}]$ and

$$\lim_{x\to\infty} v(x) = \frac{1}{2\alpha^2}, \quad \lim_{x\to\infty} \frac{v(x)}{x^2} = \frac{1}{\alpha}.$$

A formula for ξ^*

We want a nonnegative nondecreasing and adapted process ξ^* such that $X^*(t) \triangleq x + W(t) - \xi^*(t) \le \frac{1}{\sqrt{2\alpha}}$. The answer is

$$\xi^*(t) = \sup_{0 \le s \le t} [x + W(s) - \frac{1}{\sqrt{2\alpha}}]^+.$$

ξ^* is obviously nonnegative nondecreasing and adapted. The nonnegativity implies $X^*(t) \le x + W(t)$. This means that if already $x + W(t_0) \le \frac{1}{\sqrt{2\alpha}}$, then $X^*(t_0) \le \frac{1}{\sqrt{2\alpha}}$. On the other hand, if $x + W(t_0) > \frac{1}{\sqrt{2\alpha}}$, then $x + W(t_0) - \frac{1}{\sqrt{2\alpha}} = [x + W(t_0) - \frac{1}{\sqrt{2\alpha}}]^+ \le \xi^*(t_0)$. Therefore $X^*(t_0) = x + W(t_0) - \xi^*(t_0) \le \frac{1}{\sqrt{2\alpha}}$ as desired. We note that ξ^* is a process with continuous paths, and it only grows at time t if $x + W(t)$ is above $\frac{1}{\sqrt{2\alpha}}$. The oscillation properties of Brownian paths show us that furthermore the paths of ξ^* are *singularly continuous*. Recall that a real function $f(x)$ is called singular, if $f'(x) = 0$ for almost every x. It is this singularity which gives this control problem its name.

This concludes our wish-list: (1) to (7) hold. We are thus given an explicit solution-candidate for our control problem. The remaining task is to check if this candidate actually does solve the control problem.

Verification Theorem

Given the explicit (v, ξ^*) we have to show two things:

(1) $v(x) = \min_\xi I\!\!E \int_0^\infty e^{-\alpha t} X_t^2 \, dt$.

(2) The minimum is attained at ξ^*.

To do this, let ξ be *any* nondecreasing process starting at zero, $X(t) \triangleq x + W(t) - \xi(t)$. v is twice differentiable, whence Itô's rule implies

$$de^{-\alpha t} v(X(t)) = e^{-\alpha t}[-\alpha v dt + \frac{1}{2} v'' dt + v' dW_t - v' d\xi_t]$$

$$= e^{-\alpha t} I\!\!I_{\{X_t \le \frac{1}{\sqrt{2\alpha}}\}}(-X^2(t))dt + e^{-\alpha t} I\!\!I_{\{X_t > \frac{1}{\sqrt{2\alpha}}\}}(-X^2(t) + N(t))dt$$

$$+ e^{-\alpha t} v'(X(t)) dW_t - e^{-\alpha t} v'(X(t)) d\xi_t$$

for some non-negative N(t). Integrating from 0 to T and taking expectations yields

$$I\!\!E[e^{-\alpha T} v(X(T))] = v(x) - I\!\!E \int_0^T e^{-\alpha t} X^2(t) \, dt$$

$$+ I\!\!E \int_0^T e^{-\alpha t} I\!\!I_{\{X_t > \frac{1}{\sqrt{2\alpha}}\}} N(t) \, dt - I\!\!E \int_0^T e^{-\alpha t} v'(X(t)) d\xi(t).$$

The Itô integral vanished, because $e^{-\alpha t} \in [0,1]$ and $0 \ge v'(x) \ge \frac{1}{\alpha^2}[2\alpha x - \sqrt{2\alpha}]$. The process $N(t)$ is nonnegative and so

$$I\!\!E[e^{-\alpha T} v(X(T))] \ge v(x) - I\!\!E \int_0^T e^{-\alpha t} X^2(t) \, dt - I\!\!E \int_0^T e^{-\alpha t} v'(X(t)) d\xi(t)$$

$$\ge v(x) - I\!\!E \int_0^T e^{-\alpha t} X^2(t) \, dt.$$

The last step used the facts that v' is nonpositive and $d\xi$ is nonnegative (because ξ is nondecreasing). We obtain

$$v(x) \le I\!\!E \int_0^T e^{-\alpha t} X^2(t) \, dt + I\!\!E[e^{-\alpha T} v(X(T))].$$

Ideally we would like the second term to vanish as T gets large. We can do this in the following way: First we can convince ourselves that we can restrict our attention to those controls for which $I\!\!E \int_0^\infty e^{-\alpha t} X^2(t) \, dt$ is finite. This is possible, because doing nothing (i.e. $\xi = 0$) results in a finite expected cost. Doing nothing, however, may still be suboptimal: we will get no larger expected cost for the optimal ξ. Using property (5) of v we see that

$$\int_0^\infty I\!\!E e^{-\alpha t} v(X(t)) \, dt \le \int_0^\infty I\!\!E e^{-\alpha t} \left[\frac{X^2(t)}{\alpha} + \frac{1}{\alpha^2} \right] dt < \infty.$$

This implies for the positive integrand $I\!\!E e^{-\alpha t} v(X(t))$

$$\liminf_{t \to \infty} I\!\!E e^{-\alpha t} v(X(t)) = 0.$$

This enables us to choose a sequence $t_n \uparrow \infty$, such that

$$\lim_{n \to \infty} I\!\!E e^{-\alpha t_n} v(X(t_n)) = 0.$$

Using this sequence we conclude

$$v(x) \leq I\!\!E \int_0^\infty e^{-\alpha t} X^2(t)\, dt$$

for all controls ξ that have a chance to minimize the expected cost. It remains to prove that for the candidate ξ^* this inequality can't be strict. We start with the same equality that we have derived before, which must also hold for ξ^*:

$$I\!\!E[e^{-\alpha T} v(X^*(T))] = v(x) - I\!\!E \int_0^T e^{-\alpha t} X^{*2}(t)\, dt$$

$$+I\!\!E \int_0^T e^{-\alpha t} I\!\!I_{\{X_t^* > \frac{1}{\sqrt{2\alpha}}\}} N(t)\, dt - I\!\!E \int_0^T e^{-\alpha t} v'(X^*(t)) d\xi^*(t).$$

Here we wrote $X^*(t)$ for $x + W(t) - \xi^*(t)$. As for the middle integral, recall that ξ^* is designed in such a way that the event $\{X_t^* > \frac{1}{\sqrt{2\alpha}}\}$ never occurs. As for the last integral we have observed that for $X^*(t) < \frac{1}{\sqrt{2\alpha}}$ the control $\xi^*(t)$ does not grow, whence $d\xi^*(t) = 0$, and for $X^*(t) = \frac{1}{\sqrt{2\alpha}}$ we have $v'(X^*(t)) = 0$. We are left with

$$I\!\!E[e^{-\alpha T} v(X^*(T))] = v(x) - I\!\!E \int_0^T e^{-\alpha t} X^{*2}(t)\, dt.$$

All possible sources of non-strict inequalities have dissappeared. We can now let T go to infinity in a similar way as before and arrive at the desired result

$$v(x) = I\!\!E \int_0^\infty e^{-\alpha t} X^{*2}(t)\, dt,$$

which proves that our solution-candidate does in fact solve the control problem.

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