Valuation of Exotic Options Under Shortselling Constraints

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May 23, 2000

Abstract. Options with discontinuous payoffs are generally traded above their theoretical Black-Scholes prices because of the hedging difficulties created by their large delta and gamma values. A theoretical method for pricing these options is to constrain the hedging portfolio and incorporate this constraint into the pricing by computing the smallest initial capital which permits super-replication of the option. We develop this idea for exotic options, in which case the pricing problem becomes one of stochastic control. Our motivating example is a call which knocks out in the money, and explicit formulas for this and other instruments are provided.

Key words: Exotic options, super-replication, stochastic control.
JEL classification: G13
Mathematics Subject Classification (1991): 90A09, 60H30, 60G44

*Financial support by Crédit Suisse Group is gratefully acknowledged.
†Work supported by the National Science Foundation under grants DMS-95-00626 and DMS-98-02464. Part of this paper was written while this author was a visitor at the Institute for Mathematics Research, ETH Zürich, whose support is gratefully acknowledged.

1 In-the-money knock-out call

The results reported in this paper were motivated by the problem of pricing and hedging a particular exotic option, a call which knocks out in the money. More specifically, we assume a geometric Brownian motion model

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t), \quad S(0) > 0, \tag{1.1}$$

for an underlying asset, henceforth called a *stock*, even though it is often some other instrument, such as an exchange rate. The *interest rate* $r \in \mathbb{R}$, the *volatility* $\sigma > 0$ and the planning horizon $T > 0$ are assumed to be constant. The process $(W(t); 0 \le t \le T)$ is a Brownian motion under a probability measure $\mathbb{P}$ which is *risk-neutral,* i.e., is chosen so that the stock has mean rate of return r.

We introduce a *knock-out call option* whose payoff at expiration date T is

$$(S(T) - K)^+ I_{\{\max_{0\le t\le T} S(t) < B\}}, \tag{1.2}$$

where the *strike price* K and the *knock-out barrier* B satisfy $0 < K < B$ and I_A denotes the indicator of the generic event A. This call "knocks out" in the money, which makes the implementation of the Black-Scholes hedging strategy difficult, as we now explain.

If $0 \le t \le T$, $S(t) = x > 0$, and the call has not knocked out prior to time t, then the value of the call at time t is given by the risk-neutral pricing formula

$$v(t,x) \triangleq \mathbb{E}\big[e^{-r(T-t)}(S(T) - K)^+ I_{\{\max_{t\le u\le T} S(u) < B\}} \;\big|\; S(t) = x\big]. \tag{1.3}$$

The joint distribution of the drifted Brownian increment $Y = W(T) - W(t) + \theta(T-t)$ and its maximum $Z = \max_{t\le s\le T}(W(s) - W(t) + \theta(s-t))$ over the interval $[t,T]$ can be derived using Girsanov's theorem (see formula 1.1.8 of [1] or [12], Section 3.5) and is, for all $z > 0$ and $y < z$,

$$\mathbb{P}\{Y \in dy,\, Z \in dz\} = \frac{2(2z-y)}{\tau\sqrt{2\pi\tau}} \exp\left\{-\frac{(2z-y)^2}{2\tau} + \theta y - \frac{1}{2}\theta^2\tau\right\} dy\,dz, \tag{1.4}$$

where $\tau \triangleq T - t$. Let N denote the standard normal distribution function. Using formula (1.4), $v(t,x)$ can be computed explicitly: For $t \in [0,T)$ and $x \in (0,B]$,

$$\begin{aligned} v(t,x) = {} & x\big[N(b-\theta_+) - N(k-\theta_+)\big] \\ & + xe^{2b\theta_+}\big[N(b+\theta_+) - N(2b-k+\theta_+)\big] \\ & - Ke^{-r\tau}\big[N(b-\theta_-) - N(k-\theta_-)\big] \\ & - Ke^{-r\tau+2b\theta_-}\big[N(b+\theta_-) - N(2b-k+\theta_-)\big], \end{aligned} \tag{1.5}$$

where $b \triangleq \frac{1}{\sigma\sqrt{\tau}} \log\frac{B}{x}$, $k \triangleq \frac{1}{\sigma\sqrt{\tau}} \log\frac{K}{x}$ and $\theta_\pm = (\frac{r}{\sigma} \pm \frac{\sigma}{2})\sqrt{\tau}$.

Definition (1.3) implies that $v(t,B) = 0$ for $0 \le t \le T$. For $0 < x \le B$, as $t \uparrow T$, we obtain from (1.5) that $v(t,x)$ approaches the discontinuous limit $v(T,x) = (x-K)^+ I_{\{x<B\}}$. Consequently, for x near B and t near T, the "delta" $v_x(t,x)$ and "gamma" $v_{xx}(t,x)$ of this option become large in absolute value. The slope and the curvature of the dashed lines in Figure 1 illustrate

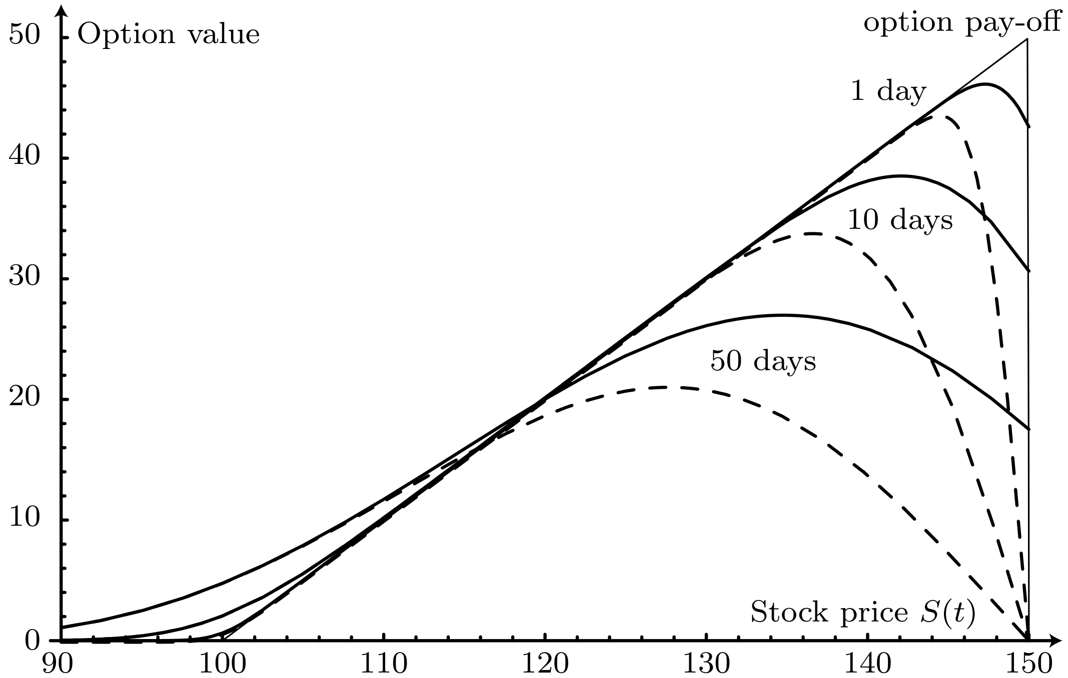

Figure 1: Upper hedging prices $v^*(0, S_0; \alpha)$ given by (4.10) for call options with strike $K = 100$, knock-out barrier $B = 150$ and maturities $T = 50/365$, $T = 10/365$ and $T = 1/365$. We used the interest rate $r = 5\%$, the volatility $\sigma = 30\%$ and the hedge-portfolio constraint $\alpha = 10$. The dashed lines show the corresponding prices given by (1.5) without the hedge-portfolio constraint (2.2).

this. As a result, a trader seeking to hedge a short position in this option will find himself taking a large short position in the underlying stock and making large adjustments to this position frequently. As a practical matter, this naive implementation of the delta hedging strategy is not possible.

A common pricing practice for such options is to *move the barrier*. If one prices and hedges the option as if the barrier were some number $B' > B$, then the dangerous region of high delta and gamma can be moved above B, and the option will knock out before the stock reaches this region. Of course, the computed price of the option increases with increasing B', and there is no clear procedure for choosing an appropriate value for B'. Furthermore, this practice necessarily prices one option at a time, rather than pricing a book of options by taking into account offsetting exposures in the book.

We propose in this paper an alternative to moving the barrier, namely, *constraining the hedging portfolio*. In Section 4 we show that this can be interpreted in terms of the transaction cost associated with liquidating a large short position (Remark 4.4), and also provides a first-order approximation to the price obtained by moving the barrier (Remark 4.3). Furthermore, the theory applies to a book of options as well as to individual options, although the computational issues for a book can be substantial. We work out a simple case of a book of two barrier options in Example 6.7.

2 Model formulation

Throughout this paper, we work within the context of the canonical probability space for Brownian motion. In particular, we take Ω to be the set of continuous functions from $[0,T]$ to $\mathbb{R}$ taking the value zero at zero, we take $\mathbb{P}$ to be Wiener measure, and we take $W(t,\omega) = \omega(t)$ for all $t \in [0,T]$ and all $\omega \in \Omega$. For $0 \le t \le T$, we denote by $\mathcal{F}^W(t)$ the σ-algebra generated by $(W(s); 0 \le s \le t)$. The σ-algebra $\mathcal{F}(T)$ is the $\mathbb{P}$-completion of $\mathcal{F}^W(T)$, and for $0 \le t \le T$, $\mathcal{F}(t)$ is the augmentation of $\mathcal{F}^W(t)$ by the $\mathbb{P}$-null sets of $\mathcal{F}(T)$. A random variable X is $\mathcal{F}(t)$-measurable if and only if there exists an $\mathcal{F}^W(t)$-measurable random variable Y with $\{X \ne Y\} \in \mathcal{F}(T)$ and $\mathbb{P}(X \ne Y) = 0$.

We introduce a contingent claim whose payoff at expiration date T is $g(S(\cdot))$. Let $C_+[0,T]$ denote the space of nonnegative continuous functions on $[0,T]$. We assume that the nonnegative function $g\colon C_+[0,T] \to [0,\infty)$ is lower semicontinuous in the supremum norm topology. The argument of g is the path of the stock price process S from date 0 to date T, and because this path is random, $g(S(\cdot))$ is a random variable on $(\Omega, \mathcal{F}(T), \mathbb{P})$.

The problem of super-replication of a short position in this option can be posed as follows. Let $X(0) > 0$ be a given nonrandom *initial wealth,* and choose an $(\mathcal{F}(t); 0 \le t \le T)$-adapted *portfolio process* $(\pi(t); 0 \le t \le T)$ and *cumulative consumption process* $(C(t); 0 \le t \le T)$. We interpret $\pi(t)$ as the proportion of wealth invested in the stock at time t (sometimes called the *gearing*). The remaining wealth is invested at interest rate r, and $C(t)$ is the amount of wealth consumed up to time t. Hence, $C(t)$ is nondecreasing, right-continuous with left limits, and $C(0) = 0$. This leads us to model the differential of wealth as

$$\begin{aligned} dX(t) &= \pi(t)X(t)\frac{dS(t)}{S(t)} + rX(t)(1-\pi(t))\,dt - dC(t) \\ &= rX(t)\,dt + \sigma\pi(t)X(t)\,dW(t) - dC(t). \end{aligned} \tag{2.1}$$

If $X(T) \ge g(S(\cdot))$ almost surely, we say that (π, C) *super-replicates* $g(S(\cdot))$ beginning with initial wealth $X(0)$.

Next, given some fixed number $\alpha \in [0,\infty)$, we impose the *portfolio constraint*

$$\pi(t) \ge -\alpha, \quad 0 \le t \le T, \text{ almost surely.} \tag{2.2}$$

The point of this constraint, in the context of the knock-out call of the previous section, is to avoid short positions which are too large relative to the value of the contingent claim being hedged. The parameter α must be chosen by the person pricing the contingent claim; in the case of the knock-out call, we interpret α in terms of a transaction cost in Remark 4.4, and this provides a guide to choosing it. If $\alpha = 0$, then short positions in the underlying are prohibited.

The *upper hedging price* of the contingent claim $g(S(\cdot))$ is defined to be

$$v(0,S(0);\alpha) \triangleq \inf\left\{ X(0) \;\middle|\; \begin{array}{l} \text{there exists a portfolio process } \pi \\ \text{satisfying (2.2) and there exists} \\ \text{a cumulative consumption process } C \\ \text{such that } X(T) \ge g(S(\cdot)) \text{ almost surely} \end{array} \right\}. \tag{2.3}$$

Cvitanić & Karatzas [5] have shown that when $v(0, S(0); \alpha)$ is finite, there exists an $X(0)$, denoted $\widehat{X}(0)$, and corresponding portfolio and consumption processes $\hat{\pi}$ and $\widehat{C}$ attaining the infimum in (2.3). We denote the corresponding wealth process by $\widehat{X}(t)$, $0 \le t \le T$. For $0 \le t < T$, we define the *upper hedging price at time* t of the contingent claim $g(S(\cdot))$ to be $\widehat{X}(t)$. The upper hedging price $\widehat{X}(t)$ generally exceeds the risk-neutral price $\mathbb{E}[e^{-r(T-t)} g(S(\cdot)) \,|\, \mathcal{F}(t)]$ because the upper hedging price includes a "reserve" to offset the portfolio constraint. During the evolution of the process, some part of this reserve might be revealed to be unnecessary. The process $\widehat{C}$ is included in the formulation of the upper hedging price so that unnecessary reserve can be removed and thus no longer included in the upper hedging price.

Cvitanić & Karatzas [5] and El Karoui & Quenez [8] have shown that the problem of computing the upper hedging price, which is a minimization problem, can be transformed to a dual maximization problem. Their results apply to path-dependent contingent claims written on multiple assets whose models may have random, time-varying volatilities, and they require only that π be constrained to lie in a closed, convex set. The dual problem is one of maximization over changes of probability measure, and in its full generality is not easy to solve. In our model, the dual problem takes the form of (2.4) below.

Theorem 2.1 (Cvitanić & Karatzas, El Karoui & Quenez) *The upper hedging price of (2.3) satisfies*

$$v(0, S(0); \alpha) = \sup_{\lambda} \mathbb{E}_\lambda \big[e^{-rT - \alpha\lambda(T)} g(S(\cdot)) \big], \tag{2.4}$$

where the supremum is over all adapted, nondecreasing, processes which are Lipschitz continuous in t, uniformly in ω, and satisfy $\lambda(0) = 0$. Here $\mathbb{E}_\lambda$ denotes expectation under the probability measures $\mathbb{P}_\lambda$ whose Radon-Nikodým derivative with respect to $\mathbb{P}$ is

$$\frac{d\mathbb{P}_\lambda}{d\mathbb{P}} = \exp\left\{ -\frac{1}{\sigma} \int_0^T \lambda'(t)\, dW(t) - \frac{1}{2\sigma^2} \int_0^T (\lambda'(t))^2 dt \right\}. \tag{2.5}$$

The supremum in (2.4) over Lipschitz continuous processes is often not attained, and Lipschitz continuity is not easily relaxed in Theorem 2.1 because of the need to define $\mathbb{P}_\lambda$ by (2.5). In this paper we shall formulate the dual problem in such a way that no change of measure is required, and we can then extend the class of processes over which the supremum in the dual problem is computed.

Broadie, Cvitanić & Soner [4] showed that in the case of a contingent claim whose payoff at expiration is a function of the final value of a single, geometric Brownian motion, the dual problem can be solved in two steps. One first computes a certain transform, which we call the *face-lift,* of the payoff function (see (2.6) below). One next prices the contingent claim whose payoff at the final time is the face-lifted version of the original payoff. One does this using the usual risk-neutral pricing formula, i.e., without regard to the portfolio constraint. For our model, the result of [4] takes the form of the next theorem. A presentation of the results of both [5] and [4] in full generality may be found in [13].

Theorem 2.2 (Broadie, Cvitanić & Soner) *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be lower semicontinuous, and suppose the contingent claim $g(S(\cdot))$ is given by $g(S(\cdot)) = \varphi(S(T))$. Define*

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda \geq 0} e^{-\alpha\lambda}\varphi(xe^{-\lambda}), \quad x \geq 0. \tag{2.6}$$

Then the upper hedging price under hedge-portfolio constraint (2.2) is given by

$$v(0, S(0); \alpha) = \mathbb{E}\big[e^{-rT}\widehat{\varphi}_\alpha(S(T))\big]. \tag{2.7}$$

The goal of this paper is to extend Theorem 2.2 to path-dependent options. The main result is that in place of the face-lifting procedure (2.6), one must solve a singular stochastic control problem. This problem can sometimes be solved by inspection, and in particular, such a solution is possible for the knock-out call of the previous section. The solution of the stochastic control problem leads directly to a formula for the upper hedging price, in the spirit of (2.7).

The present paper is more general than [4] in that it allows path-dependent options, but more special in that the only portfolio constraint considered here is (2.2), whereas [4] permits a general convex constraint on π. There appears to be no insurmountable obstacle to working out a theory along the lines of the present paper for the more general constraint.

The role of upper hedging prices in the presence of stochastic volatility and/or transaction costs is studied in [2], [6], [7], [15]. Gamma constraints are treated in [14]. Lower hedging prices are introduced in [10], and [11] treats perpetual American options using similar methodology. Classical Black-Scholes prices for a large number of exotic options are provided by Zhang [17].

3 Dual problem as singular stochastic control

We wish to convert the computation of the supremum on the right-hand side of (2.4) to a singular stochastic control problem. Toward this end, we let $(W(t), \mathcal{F}(t); 0 \leq t \leq T)$ be the canonical Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}(T), \mathbb{P})$ of the previous section, and we denote

$$\begin{aligned}\mathcal{C} \triangleq \big\{\lambda; \lambda \text{ is an } \{\mathcal{F}(t); 0 \leq t \leq T\}\text{-adapted,} \\ \text{nondecreasing, continuous process with } \lambda(0) = 0\big\}.\end{aligned} \tag{3.1}$$

One result of this paper is the following.

Theorem 3.1 *Let g be a nonnegative, lower-semicontinuous function defined on $C_+[0,T]$. The upper hedging price for the contingent claim with payoff $g(S)$ at expiration date T and hedge-portfolio constraint (2.2) is*

$$v(0, S(0); \alpha) = \sup_{\lambda \in \mathcal{C}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)} g(Se^{-\lambda})\big], \tag{3.2}$$

where

$$S(t) = S(0)\exp\big(\sigma W(t) + \mu(t)\big), \quad 0 \leq t \leq T, \tag{3.3}$$

with $\mu(t) \triangleq \big(r - \frac{1}{2}\sigma^2\big)t$ is the solution of (1.1).

The problem of maximizing $\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}g(Se^{-\lambda})\big]$ over all $\lambda \in \mathcal{C}$ is one of stochastic control. In the examples we shall see that there is often a sequence of processes $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathcal{C}$ for which

$$\lim_{n\to\infty} \mathbb{E}\big[e^{-rT-\alpha\lambda_n(T)}g(Se^{-\lambda_n})\big] = v(0, S(0); \alpha)$$

and the limit λ of the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is a singularly continuous process; hence the characterization of the right-hand side of (3.2) as a singular stochastic control problem. However, the limiting λ can fail to obtain the supremum in (3.2) because g is lower semicontinuous rather than upper semicontinuous; lower semicontinuity is needed for the proof of Theorem 3.1 (see Lemma 7.2).

The difference between Theorems 2.1 and 3.1 is that whereas the former requires a maximization over changes of measure, the latter allows one to maximize over processes $\lambda \in \mathcal{C}$, always computing expectations using the same operator $\mathbb{E}$. Of course, one can use the Radon-Nikodým derivative $d\mathbb{P}_\lambda/d\mathbb{P}$ to rewrite the right-hand side of (2.4) as an expectation under the expectation operator corresponding to $\mathbb{P}$, but the presence of the Radon-Nikodým derivative in the resulting stochastic control problem complicates it considerably. As we shall see in the examples, the stochastic control problem of (3.2) can often be solved by inspection. The proof of Theorem 3.1 is given in Section 7.

4 Constrained in-the-money knock-out call

For the in-the-money knock-out call of Section 1, the function g is

$$g(y) \triangleq \big(y(T) - K\big)^+ I_{\{\max_{0\le t\le T} y(t) < B\}}, \quad y \in C_+[0, T]. \tag{4.1}$$

We have chosen to write the set $\{\max_{0\le t\le T} y(t) < B\}$ with the strict inequality so that g will be lower semicontinuous. For geometric Brownian motion (3.3), the probability of reaching a barrier is the same as the probability of crossing the same barrier, so the contingent claim defined by

$$g^*(y) \triangleq \big(y(T) - K\big)^+ I_{\{\max_{0\le t\le T} y(t) \le B\}}, \quad y \in C_+[0, T], \tag{4.2}$$

has the same upper hedging price.

We consider the problem of maximization of

$$\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(S_\lambda(T) - K\big)^+ I_{\{M_\lambda(T) < B\}}\big], \tag{4.3}$$

where

$$S_\lambda(t) \triangleq S(t)e^{-\lambda(t)}, \quad M_\lambda(t) \triangleq \max_{0\le u\le t} S_\lambda(u), \tag{4.4}$$

and $0 < S(0) < B$. The maximization is over processes $\lambda \in \mathcal{C}$. To find the maximal value of (4.3) it is clear that one should choose the nondecreasing process λ so that $M_\lambda(T)$ is strictly less than B. On the other hand, one should not have λ be any larger than necessary because λ appears in both the discount term $e^{-rT-\alpha\lambda(T)}$ and as a discount in the formula for S_λ. If g were given by (4.2), the maximizing λ would be that nondecreasing process which causes reflection of S_λ at the barrier B, i.e.,

$$\lambda^*(t) \triangleq \max_{0\le u\le t}\big(\log S(u) - \log B\big)^+. \tag{4.5}$$

Since g is dominated by g^*, we have

$$v(0,S(0);\alpha) \le \mathbb{E}\big[e^{-rT-\alpha\lambda^*(T)}\big(S_{\lambda^*}(T)-K\big)^+\big]. \tag{4.6}$$

But with g given by (4.1), it is still possible to choose a sequence of barriers $\{B_n\}_{n=1}^\infty$ converging up to B but always strictly less than B and then take the sequence of processes $\{\lambda_n\}_{n=1}^\infty$ for which λ_n causes reflection at B_n. Then $\lambda_n(T)\downarrow\lambda^*(T)$ and therefore $S_{\lambda_n}(T)\uparrow S_{\lambda^*}(T)$ as $n\to\infty$. By the bounded convergence theorem,

$$\begin{aligned} v(0,S(0);\alpha) &\ge \limsup_{n\to\infty}\,\mathbb{E}\big[e^{-rT-\alpha\lambda_n(T)}\big(S_{\lambda_n}(T)-K\big)^+\big]\\ &= \mathbb{E}\big[e^{-rT-\alpha\lambda^*(T)}\big(S_{\lambda^*}(T)-K\big)^+\big]. \end{aligned} \tag{4.7}$$

These considerations have led us to the following corollary of Theorem 3.1.

Corollary 4.1 *For $0\le t\le T$ and $0<x\le B$, define*

$$v^*(t,x;\alpha) \triangleq \mathbb{E}\big[e^{-r(T-t)-\alpha(\lambda^*(T)-\lambda^*(t))}\big(S_{\lambda^*}(T)-K\big)^+ \;\big|\; S_{\lambda^*}(t)=x\big]. \tag{4.8}$$

Let $t\in[0,T]$ be given, and assume that $S(t)=x$. Then the upper hedging price at time t of the in-the-money knock-out call of Section 1 is

$$v(t,x;\alpha) = v^*(t,x;\alpha)I_{\{x<B\}}, \tag{4.9}$$

and for $t\in[0,T)$ the function $v^*(t,x;\alpha)$ can be computed (with removable singularities at $\alpha=2r/\sigma^2$ and $\alpha=-1+2r/\sigma^2$ in the case $2r\ge\sigma^2$) to be

$$\begin{aligned} & x\Big[N(b-\theta_+)-N(k-\theta_+)\\ &\quad + e^{\frac12 s(s-2\theta_+)}\left\{e^{sb}N(-b+\theta_+-s)-e^{sk}N(-k+\theta_+-s)\right\}\Big]\\ &+\frac{sxe^{2b\theta_+}}{s-2\theta_+}\Big[N(b+\theta_+)-N(\ell+\theta_+)\\ &\quad + e^{\frac12 s(s-2\theta_+)}\big\{e^{(s-2\theta_+)b}N(-b+\theta_+-s)-e^{(s-2\theta_+)\ell}N(-\ell+\theta_+-s)\big\}\Big]\\ &- Ke^{-r\tau}\Big[N(b-\theta_-)-N(k-\theta_-)\\ &\quad + e^{\frac12\tilde s(\tilde s-2\theta_-)}\big\{e^{\tilde s b}N(-b+\theta_--\tilde s)-e^{\tilde s k}N(-k+\theta_--\tilde s)\big\}\Big]\\ &-\frac{\tilde sKe^{-r\tau+2b\theta_-}}{\tilde s-2\theta_-}\Big[N(b+\theta_-)-N(\ell+\theta_-)\\ &\quad + e^{\frac12\tilde s(\tilde s-2\theta_-)}\big\{e^{(\tilde s-2\theta_-)b}N(-b+\theta_--\tilde s)-e^{(\tilde s-2\theta_-)\ell}N(-\ell+\theta_--\tilde s)\big\}\Big], \end{aligned} \tag{4.10}$$

where we have used the abbreviations $\tau = T - t$ *and*

$$\begin{aligned} b &= \frac{1}{\sigma\sqrt{\tau}} \log \frac{B}{x}, & k &= \frac{1}{\sigma\sqrt{\tau}} \log \frac{K}{x}, & \theta_\pm &= \Big(\frac{r}{\sigma} \pm \frac{\sigma}{2}\Big)\sqrt{\tau}, \\ \ell &= 2b - k, & s &= (1+\alpha)\sigma\sqrt{\tau}, & \tilde{s} &= \alpha\sigma\sqrt{\tau}. \end{aligned}$$

PROOF: Theorem 3.1 and the argument preceding the statement of the corollary show that $v^*(0, S(0); \alpha)$ is the upper hedging price of the knock-out call for $0 < S(0) < B$. For $S(0) = B$, the call is knocked out at inception, and hence has upper hedging price zero. This establishes (4.9) when $t = 0$. For other values of t, one can verify the formula by a translation of the initial condition. Equation (4.10) is obtained by direct calculation using (1.4). ♢

It is instructive to construct the short-position hedge. Formula (4.10) with $v^*(T, x; \alpha) = (x - K)^+$ shows that $v^*(t, x; \alpha)$ is continuous on $[0, T] \times (0, B]$ and smooth on $[0, T) \times (0, B]$.

Let S be the underlying geometric Brownian motion given by (3.3). Then S_{λ^*} is a Markov process and

$$dS_{\lambda^*}(t) = S_{\lambda^*}(t)[r\,dt + \sigma\,dW(t) - d\lambda^*(t)]. \tag{4.11}$$

Moreover,

$$e^{-rt-\alpha\lambda^*(t)} v^*(t, S_{\lambda^*}(t); \alpha) = \mathbb{E}\big[e^{-rT-\alpha\lambda^*(T)}\big(S_{\lambda^*}(T) - K\big)^+ \mid \mathcal{F}(t)\big]. \tag{4.12}$$

We compute the differential using Itô's formula

$$\begin{aligned} &d\big(e^{-rt-\alpha\lambda^*(t)} v^*(t, S_{\lambda^*}(t); \alpha)\big) \\ &\quad = e^{-rt-\alpha\lambda^*(t)}\Big[-(\alpha v^* + S_{\lambda^*} v^*_x)\,d\lambda^* \\ &\qquad\qquad + \Big(-rv^* + v^*_t + rS_{\lambda^*} v^*_x + \frac{1}{2}\sigma^2 S^2_{\lambda^*} v^*_{xx}\Big)dt + \sigma S_{\lambda^*} v^*_x\,dW\Big]. \end{aligned}$$

But the right-hand side of (4.12) is a martingale, which implies

$$\begin{aligned} \big[\alpha v^*(t, S_{\lambda^*}(t); \alpha) + S_{\lambda^*}(t)\, v^*_x(t, S_{\lambda^*}(t); \alpha)\big]\,d\lambda^*(t) = 0, \\ \Big[-r v^*(t, S_{\lambda^*}(t); \alpha) + v^*_t(t, S_{\lambda^*}(t); \alpha) + rS_{\lambda^*}(t)\, v^*_x(t, S_{\lambda^*}(t); \alpha) \\ + \frac{1}{2}\sigma^2 S^2_{\lambda^*}(t)\, v^*_{xx}(t, S_{\lambda^*}(t); \alpha)\Big]dt = 0, \end{aligned}$$

i.e., for $0 \le t < T$ and $0 < x \le B$,

$$\alpha v^*(t, B; \alpha) + B v^*_x(t, B; \alpha) = 0, \tag{4.13}$$

$$v^*_t(t, x; \alpha) + rx v^*_x(t, x; \alpha) + \frac{1}{2}\sigma^2 x^2 v^*_{xx}(t, x; \alpha) = r v^*(t, x; \alpha). \tag{4.14}$$

One can also obtain (4.13) and (4.14) by direct, albeit tedious, computation beginning with (4.10).

It can be verified by direct computation that because v^* satisfies the Black-Scholes partial differential equation (4.14) in the region $[0,T)\times(0,B]$, the function xv_x^* does also; just differentiate (4.14) with respect to x and identify the resulting terms as the partial derivatives of xv_x^*. It follows that $\alpha v^* + xv_x^*$ satisfies the Black-Scholes partial differential equation. But $\alpha v^*(t,x;\alpha)+xv_x^*(t,x;\alpha)$ is nonnegative for $t=T$ and $0<x<B$, $x\neq K$, and this function is zero on the upper barrier $x=B$ for $0\leq t<T$ (see (4.13)). It follows from the maximum principle (or by regarding $\alpha v^* + xv_x^*$ as the price of a knock-out option with nonnegative payoff upon expiration) that

$$\alpha v^*(t,x;\alpha)+xv_x^*(t,x;\alpha)\geq 0,\quad 0\leq t\leq T,\, 0<x\leq B. \tag{4.15}$$

Now suppose $0<S(0)<B$, and define $\Theta \triangleq \inf\{t\geq 0;\, S(t)=B\}$ to be the time of knock-out; we allow the possibility that $\Theta>T$, i.e., knock-out does not occur before expiration. Let us begin with initial capital $\widehat{X}(0)=v^*(0,S(0);\alpha)$ and use the portfolio process

$$\hat{\pi}(t)\triangleq \frac{S(t)v_x^*(t,S(t);\alpha)}{v^*(t,S(t);\alpha)}I_{\{t\leq T\wedge\Theta\}},\quad 0\leq t\leq T, \tag{4.16}$$

(see Figure 2) and consumption process

$$\widehat{C}(t)\triangleq v^*(\Theta,B;\alpha)I_{\{\Theta\leq t\leq T\}},\quad 0\leq t\leq T. \tag{4.17}$$

Relation (4.15) shows that $\hat{\pi}(t)\geq -\alpha$ for all $t\in[0,T]$. The cumulative consumption process $\widehat{C}$ is identically zero until the option knocks out, at which time it has a positive jump; see Figure 1 for the jump size.

Equation (4.14) shows that for $0\leq t<\Theta\wedge T$,

$$dv^*(t,S(t);\alpha)=rv^*(t,S(t);\alpha)\,dt+\sigma\hat{\pi}(t)v^*(t,S(t);\alpha)\,dW(t).$$

Comparison to (2.1) shows that for $t<\Theta\wedge T$, $v^*(t,S(t);\alpha)=\widehat{X}(t)$, the wealth process corresponding to $\widehat{X}(0)$, $\hat{\pi}$ and $\widehat{C}$. If $\Theta\leq T$, then $\lim_{t\uparrow\Theta}\widehat{X}(t)=v^*(\Theta,B;\alpha)$ and $\widehat{X}(\Theta)=\lim_{t\uparrow\Theta}\widehat{X}(t)-C(\Theta)=0$. For $\Theta<t\leq T$, we also have $\widehat{X}(t)=0$. In general,

$$\widehat{X}(t)=v(t\wedge\Theta,S(t\wedge\Theta);\alpha),\quad 0\leq t\leq T, \tag{4.18}$$

where v is defined by (4.9). In particular, $\widehat{X}(T)=v(T\wedge\Theta,S(T\wedge\Theta);\alpha)=(S(T)-K)^+I_{\{\Theta>T\}}$, i.e., we have hedged a short position in the option in a manner which respects the portfolio constraint $\hat{\pi}\geq-\alpha$.

Remark 4.2 If the knock-out call payoff were given by g^* of (4.2) rather than g of (4.1), the maximum of the quantity analogous to (4.3) would be attained by λ^* of (4.5). For $0<S(0)<B$, this maximum would be the upper hedging price and this replacement would simplify the discussion preceding Corollary 4.1. However, for $S(0)=B$, this would not give the upper hedging price. If $S(0)=B$ the option is certain to knock out and the upper hedging price is zero, as is the maximum of the quantity in (4.3). However, if we replace g by g^*, the maximum of the quantity analogous to (4.3) is strictly positive, and in fact is $v^*(0,B;\alpha)$.

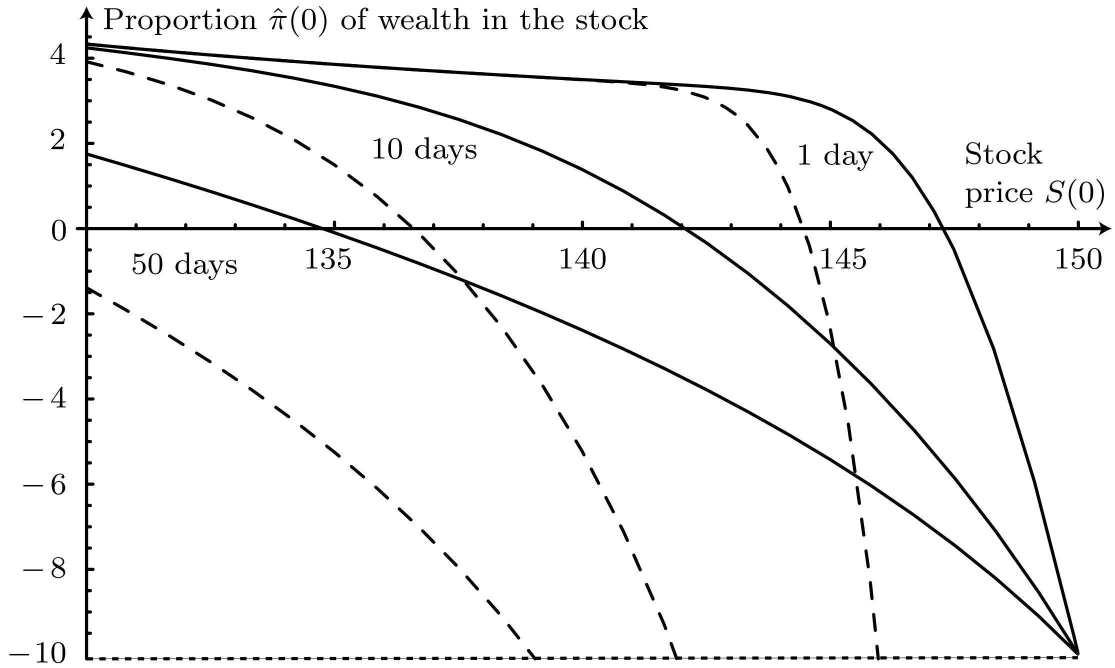

Figure 2: Proportions $\hat{\pi}(0)$ of the wealth in the underlying stock, calculated with (4.16) and (4.10), to super-replicate the in-the-money knock-out call options with parameters as in Figure 1. Note that the constraint $\pi_0 \geq -\alpha$ with $\alpha = 10$ is satisfied. The three dashed curves show the corresponding proportions $S_0 v_x(0, S(0))/v(0, S_0)$ with $v(0, S(0))$ given by (1.5) without the hedge-portfolio constraint (2.2). Note that these proportions are not bounded from below.

Remark 4.3 (Interpretation as moving the barrier) A common practical method for dealing with up-and-out call options which knock out in the money is to price and hedge the option as if the barrier were at some level B' strictly greater than the contractual barrier B. The resulting pricing function is continuous on $[0, T] \times (0, B']$, satisfies the Black-Scholes partial differential equation on $[0, T) \times (0, B']$, is zero at the barrier B', and agrees with the call payoff $(x - K)^+$ at the expiration time T. Our function $v^*(t, x; \alpha)$ is strictly positive at $x = B$. For $\alpha > 0$ we may extrapolate it linearly above this point so that it is continuously differentiable by the formula

$$v^*(t, B; \alpha) + (x - B)v_x^*(t, B; \alpha), \quad x \geq B. \tag{4.19}$$

Because of (4.13), this linear extrapolation takes the value zero at $x = \left(1 + \frac{1}{\alpha}\right) B$, independently of t. Consequently, $v^*(t, x; \alpha)$ may be regarded as an approximation to the option price obtained by moving the barrier to $B' = \left(1 + \frac{1}{\alpha}\right) B$.

Remark 4.4 (Interpretation as transaction cost) The Black-Scholes formula is based on the assumption that the bid-ask spread does not play a significant role in option hedging. A trader who hedges a short position in the knock-out option of this section can be left with a large short position in the

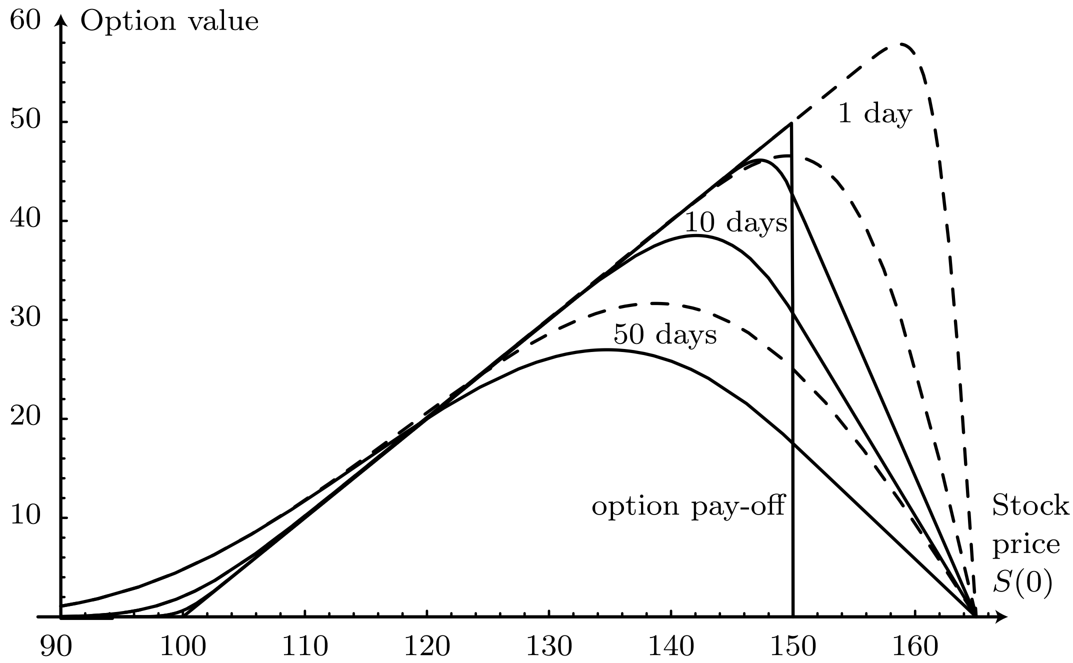

Figure 3: Upper hedging prices $v^*(0, S(0), \alpha)$ of the in-the-money knock-out call options from Figure 1. Prices are extrapolated linearly and continuously differentiably beyond the barrier $B = 150$ using (4.19). The dashed curves show the prices calculated via (1.5) without hedge-portfolio constraint (2.2) but a barrier moved to $B' = B(1+1/\alpha) = 165$. For applications, only the prices for $S(0) < B = 150$ are relevant.

underlying stock when the option knocks out, and covering this position can entail a significant transaction cost. Let us suppose the trader prices the option according to a function $v(t, x)$ which is continuous in $[0, T] \times (0, B]$, satisfies the Black-Scholes partial differential equation in $[0, T) \times (0, B]$, and agrees with the call payoff $(x - K)^+$ at the expiration time T. Using the "delta-hedging strategy" to hedge a short position, the trader will hold $v_x(t, x)$ shares of stock at time t if the stock price is x, and upon knock-out of the option, will be left with a position $v_x(t, B)$ in the stock valued at $|Bv_x(t, B)|$. Suppose $v_x(t, B)$ is negative and it requires $-\left(1 + \frac{1}{\alpha}\right) Bv_x(t, B)$ with $\alpha > 0$ to cover this short position. The total hedging portfolio value is $v(t, B)$, and since wealth invested in stock is $Bv_x(t, B)$, the wealth invested in the money market must be $v(t, B) - Bv_x(t, B)$. The money market position is exactly what is needed to cover the short stock position, taking the transaction cost into account, if and only if the equation

$$v(t, B) - Bv_x(t, B) = -\Big(1 + \frac{1}{\alpha}\Big) Bv_x(t, B)$$

holds. This is equivalent to $\alpha v(t, B) + Bv_x(t, B) = 0$ for $0 \leq t < T$, which is condition (4.13) satisfied by $v^*(t, x; \alpha)$. Together with the conditions already specified on v, this uniquely determines v, and ensures that $v(t, x) = v^*(t, x; \alpha)$.

5 The dual problem as impulsive control

Let $0 < t_1 < t_2 < \cdots < t_I \le T$ be fixed dates. For the examples of Section 6, it is helpful to generalize Theorem 3.1 to right-continuous λ with possible jumps only at these dates. We denote by $R[0,T]$ the set of nondecreasing functions λ defined on $[0,T]$, continuous on $[0,T]\setminus\{t_1,\ldots,t_I\}$, right-continuous at $t_1,\ldots,t_I$, and with $\lambda(0)=0$. We then define

$$\mathcal{R} \triangleq \big\{\lambda;\ \lambda \text{ is an } \{\mathcal{F}(t)\}_{0\le t\le T}\text{-adapted process with paths in } R[0,T]\big\}. \tag{5.1}$$

A function in $R[0,T]$ can be regarded as the cumulative distribution function of a measure on $[0,T]$. The measures corresponding to a sequence $\{\lambda_n\}_{n=1}^\infty$ in $R[0,T]$ converge weakly to the measure with cumulative distribution function $\lambda \in R[0,T]$ if and only if $\lambda_n(t) \to \lambda(t)$ at every continuity point t of λ and for $t = T$. Because the weak topology on measures can be metrized (see [16]), there is a metric d_w on $R[0,T]$ satisfying

$$\lambda_n(t) \to \lambda(t) \text{ for every continuity point } t \text{ of } \lambda \text{ and } t = T \iff d_w(\lambda_n,\lambda) \to 0.$$

We may define a stronger metric d on $R[0,T]$ by

$$d(\eta,\lambda) = d_w(\eta,\lambda) + \sum_{i=1}^{I} |\eta(t_i) - \lambda(t_i)|.$$

Then for every sequence $\{\lambda_n\}_{n=1}^\infty$ in $R[0,T]$ and $\lambda \in R[0,T]$,

$$\lambda_n(t) \to \lambda(t)\ \forall\, t \in [0,T] \iff d(\lambda_n,\lambda) \to 0, \tag{5.2}$$

i.e., d metrizes pointwise convergence in $R[0,T]$.

Remark 5.1 If λ in (5.2) is continuous, pointwise convergence of λ_n to λ implies uniform convergence. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|t-s| \le \delta$ implies $|\lambda(t) - \lambda(s)| \le \varepsilon$. Choose $0 = s_0 < s_1 < \cdots < s_K = T$ such that $s_{k+1} - s_k \le \delta$ for all k. If $\lambda_n \to \lambda$ pointwise, we may choose N so that $|\lambda_n(s_k) - \lambda(s_k)| \le \varepsilon$ for every $n \ge N$ and every k. Given $t \in [0,T]$, we choose k so that $s_k \le t \le s_{k+1}$, and then for all $n \ge N$,

$$\begin{aligned}
|\lambda_n(t) - \lambda(t)| &\le \lambda_n(t) - \lambda_n(s_k) + |\lambda_n(s_k) - \lambda(s_k)| + |\lambda(s_k) - \lambda(t)| \\
&\le \lambda_n(s_{k+1}) - \lambda_n(s_k) + 2\varepsilon \\
&\le |\lambda_n(s_{k+1}) - \lambda(s_{k+1})| + |\lambda(s_{k+1}) - \lambda(s_k)| \\
&\qquad + |\lambda(s_k) - \lambda_n(s_k)| + 2\varepsilon \\
&\le 5\varepsilon.
\end{aligned}$$

More generally, suppose $\lambda \in R[0,T]$ is not necessarily continuous. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $R[0,T]$ converging pointwise to λ, and let $\delta > 0$ be given. Then $\{\lambda_n\}_{n=1}^\infty$ converges uniformly to λ on $[0,T] \setminus \bigcup_{i=1}^{I}(t_i - \delta, t_i)$. Indeed, the argument in the previous paragraph shows uniform convergence on each connected component of this set, and since there are only finitely many such components, the convergence is uniform on the whole set. $\diamondsuit$

The function g is defined on $C_+[0,T]$, the space of nonnegative continuous functions on $[0,T]$. We extend it to $C_+[0,T] \times R[0,T]$ by the definition

$$g_*(y,\lambda) \triangleq \inf\left\{ \liminf_{n\to\infty} g\left(ye^{-\lambda_n}\right) \middle| \begin{array}{l} \{\lambda_n\}_{n=1}^\infty \text{ is a sequence in } R[0,T] \\ \cap\, C_+[0,T] \text{ converging pointwise to } \lambda \end{array} \right\},$$
$$y \in C_+[0,T],\ \lambda \in R[0,T]. \quad (5.3)$$

Proposition 5.2 *Suppose $g\colon C_+[0,T] \to [0,\infty)$ is of the form*

$$g(y) = \varphi(y(t_1),\dots,y(t_I), m(y), M(y), A(y)),$$

where

$$m(y) \triangleq \inf_{0\le t\le T} y(t), \quad M(y) \triangleq \sup_{0\le t\le T} y(t), \quad A(y) \triangleq \frac{1}{T}\int_0^T y(t)\,dt,$$

and $\varphi\colon \mathbb{R}^{I+3} \to [0,\infty)$ is a lower-semicontinuous function which is jointly left-continuous in its last three arguments. Then for $y \in C_+[0,T]$ and $\lambda \in R[0,T]$,

$$g_*(y,\lambda) = \varphi\Big(y(t_1)e^{-\lambda(t_1)},\dots,y(t_I)e^{-\lambda(t_I)}, m\big(ye^{-\lambda}\big), M\big(ye^{-\lambda}\big), A\big(ye^{-\lambda}\big)\Big).$$

PROOF: We claim that for fixed $y \in C_+[0,T]$, the mappings $\lambda \mapsto m(ye^{-\lambda})$, $\lambda \mapsto M(ye^{-\lambda})$ and $\lambda \mapsto A(ye^{-\lambda})$ are continuous from $R[0,T]$ to $[0,\infty)$. Indeed, fix $y \in C_+[0,T]$ and suppose $\{\lambda_n\}_{n=1}^\infty$ converges pointwise to λ. Let $\varepsilon > 0$ be given and choose $\delta > 0$ so that $|y(s) - y(t)| \le \varepsilon$ whenever $|s-t| \le \delta$. For the sake of notational simplicity we define $t_0 \triangleq 0$ and assume that $t_I = T$. We may assume without loss of generality that $\delta < \min_{1\le i\le I}(t_i - t_{i-1})$. Because of Remark 5.1, we may choose $N(\varepsilon)$ so large that whenever $n \ge N(\varepsilon)$, we have

$$|\lambda_n(t) - \lambda(t)| \le \varepsilon \quad \forall\, t \in \textstyle\bigcup_{i=1}^I [t_{i-1}, t_i - \delta].$$

For these $t \in \bigcup_{i=1}^I [t_{i-1}, t_i - \delta]$, we have

$$\begin{aligned} \left|y(t)e^{-\lambda_n(t)} - y(t)e^{-\lambda(t)}\right| &\le M(y)e^{-\lambda(t)}\left|e^{\lambda(t)-\lambda_n(t)} - 1\right| \\ &\le M(y)(e^\varepsilon - 1). \end{aligned} \quad (5.4)$$

On the other hand, for $i \in \{1,\dots,I\}$ and $t \in [t_i - \delta, t_i]$, we have

$$\begin{aligned} y(t)e^{-\lambda_n(t)} &\le y(t)e^{-\lambda_n(t_i-\delta)} \\ &\le |y(t) - y(t_i-\delta)|\, e^{-\lambda_n(t_i-\delta)} + y(t_i-\delta)e^{-\lambda_n(t_i-\delta)} \\ &\le \varepsilon + y(t_i-\delta)e^{-\lambda(t_i-\delta)} + y(t_i-\delta)\left|e^{-\lambda_n(t_i-\delta)} - e^{-\lambda(t_i-\delta)}\right| \\ &\le \varepsilon + M\big(ye^{-\lambda}\big) + M(y)(e^\varepsilon - 1). \end{aligned}$$

Combining this inequality with (5.4), we see that

$$M\big(ye^{-\lambda_n}\big) \le \varepsilon + M\big(ye^{-\lambda}\big) + M(y)(e^\varepsilon - 1). \quad (5.5)$$

Similarly,

$$\begin{aligned}
y(t)e^{-\lambda(t)} &\leq y(t)e^{-\lambda(t_i-\delta)} \\
&\leq |y(t) - y(t_i - \delta)|\, e^{-\lambda(t_i-\delta)} + y(t_i - \delta)e^{-\lambda(t_i-\delta)} \\
&\leq \varepsilon + y(t_i - \delta)e^{-\lambda_n(t_i-\delta)} + y(t_i - \delta)\big|e^{-\lambda(t_i-\delta)} - e^{-\lambda_n(t_i-\delta)}\big| \\
&\leq \varepsilon + M\big(ye^{-\lambda_n}\big) + M(y)(e^{\varepsilon} - 1).
\end{aligned}$$

We may combine this with (5.4) to obtain

$$M\big(ye^{-\lambda}\big) \leq \varepsilon + M\big(ye^{-\lambda_n}\big) + M(y)(e^{\varepsilon} - 1). \tag{5.6}$$

We conclude that for $n \geq N(\varepsilon)$, the inequality

$$\big|M\big(ye^{-\lambda_n}\big) - M\big(ye^{-\lambda}\big)\big| \leq \varepsilon + M(y)(e^{\varepsilon} - 1)$$

holds, and hence the mapping $\lambda \mapsto M\big(ye^{-\lambda}\big)$ is continuous. Similar arguments show the continuity of $\lambda \mapsto m\big(ye^{-\lambda}\big)$. The continuity of $\lambda \mapsto A\big(ye^{-\lambda}\big)$ follows from the dominated convergence theorem.

Now let $y \in C_+[0,T]$ and $\lambda \in R[0,T]$ be given, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $R[0,T] \cap C_+[0,T]$ converging pointwise to λ. Because $\lim_{n\to\infty} m\big(ye^{\lambda_n}\big) = m\big(ye^{-\lambda}\big)$, $\lim_{n\to\infty} M\big(ye^{\lambda_n}\big) = M\big(ye^{-\lambda}\big)$, $\lim_{n\to\infty} A\big(ye^{\lambda_n}\big) = A\big(ye^{-\lambda}\big)$ and φ is lower semicontinuous, we have

$$\begin{aligned}
&\varphi\Big(y(t_1)e^{-\lambda(t_1)}, \ldots, y(t_I)e^{-\lambda(t_I)}, m\big(ye^{-\lambda}\big), M\big(ye^{-\lambda}\big), A\big(ye^{-\lambda}\big)\Big) \\
&\quad\leq \liminf_{n\to\infty} \varphi\Big(y(t_1)e^{-\lambda_n(t_1)}, \ldots, y(t_I)e^{-\lambda_n(t_I)}, m\big(ye^{-\lambda_n}\big), M\big(ye^{-\lambda_n}\big), A\big(ye^{-\lambda_n}\big)\Big) \\
&\quad= \liminf_{n\to\infty} g\big(ye^{-\lambda_n}\big).
\end{aligned}$$

Taking the infimum of the right-hand side over sequences $\{\lambda_n\}_{n=1}^{\infty}$ converging pointwise to λ, we conclude that

$$\varphi\Big(y(t_1)e^{-\lambda(t_1)}, \ldots, y(t_I)e^{-\lambda(t_I)}, m\big(ye^{-\lambda}\big), M\big(ye^{-\lambda}\big), A\big(ye^{-\lambda}\big)\Big) \leq g_*\big(y, \lambda\big).$$

To obtain the reverse inequality, we choose $\lambda_n \in R[0,T] \cap C_+[0,T]$ so that $\lambda_n(t_i) = \lambda(t_i)$ for $i = 1, \ldots, I$ and $\lambda_n(t) \downarrow \lambda(t)$ for every $t \in [0,T]$ as $n \to \infty$. Then $ye^{-\lambda_n} \uparrow ye^{-\lambda}$ pointwise. The joint left-continuity of φ in its last three variables implies

$$\begin{aligned}
&\varphi\Big(y(t_1)e^{-\lambda(t_1)}, \ldots, y(t_I)e^{-\lambda(t_I)}, m\big(ye^{-\lambda}\big), M\big(ye^{-\lambda}\big), A\big(ye^{-\lambda}\big)\Big) \\
&\quad= \lim_{n\to\infty} \varphi\Big(y(t_1)e^{-\lambda_n(t_1)}, \ldots, y(t_I)e^{-\lambda_n(t_I)}, m\big(ye^{-\lambda_n}\big), M\big(ye^{-\lambda_n}\big), A\big(ye^{-\lambda_n}\big)\Big) \\
&\quad= \lim_{n\to\infty} g\big(ye^{-\lambda_n}\big) \\
&\quad\geq g_*(y, \lambda).
\end{aligned}$$

$\diamondsuit$

Theorem 5.3 *Let g be a nonnegative, lower-semicontinuous function defined on $C_+[0,T]$. The upper hedging price for the contingent claim with payoff $g(S)$ at expiration date T and hedge-portfolio constraint* (2.2) *is*

$$v(0, S(0); \alpha) = \sup_{\lambda \in \mathcal{R}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)} g_*(S, \lambda)\big], \tag{5.7}$$

where the geometric Brownian motion S is given by (3.3).

The proof of Theorem 5.3 is given in Section 8.

Remark 5.4 Theorem 5.3 leads immediately to an alternate proof of Theorem 2.2 (Broadie, Cvitanić & Soner). Let $\varphi\colon [0,\infty) \to [0,\infty)$ be lower semicontinuous, and define $g\colon C_+[0,T] \to [0,\infty)$ by $g(y) = \varphi(y(T))$. In the definition of $\mathcal{R}$, take $I = 1$ and $t_1 = T$, i.e., the only allowed discontinuity for processes in $\mathcal{R}$ is at time T. Proposition 5.2 implies that $g_*(y, \lambda) = \varphi\big(y(T)e^{-\lambda(T)}\big)$. According to Theorem 5.3, the upper hedging price is

$$v(0, S(0); \alpha) = \sup_{\lambda \in \mathcal{R}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)} \varphi\big(S(T)e^{-\lambda(T)}\big)\big], \tag{5.8}$$

which is obviously bounded above by $\mathbb{E}\big[e^{-rT}\widehat{\varphi}_\alpha(S(T))\big]$ (see (2.6) for notation). On the other hand, a selection theorem due to Freedman [9] (see, e.g., [3], Proposition 7.34) asserts that for each $\varepsilon > 0$ there is a Borel measurable function $\psi_\varepsilon\colon [0,\infty) \to [0,\infty)$ satisfying

$$e^{-\alpha\psi_\varepsilon(x)}\varphi\big(xe^{-\psi_\varepsilon(x)}\big) \geq \begin{cases} \widehat{\varphi}_\alpha(x) - \varepsilon & \text{if } \widehat{\varphi}_\alpha(x) < \infty, \\ 1/\varepsilon & \text{if } \widehat{\varphi}_\alpha(x) = \infty. \end{cases}$$

Taking $\lambda(t) = I_{\{t=T\}}\psi_\varepsilon(S(T))$, we conclude from (5.8) that

$$v(0, S(0); \alpha) \geq -\varepsilon + \mathbb{E}\big[I_{\{\widehat{\varphi}_\alpha(S(T))<\infty\}} e^{-rT}\widehat{\varphi}_\alpha(S(T))\big] + \frac{e^{-rT}}{\varepsilon}\,\mathbb{P}\{\widehat{\varphi}_\alpha(S(T)) = \infty\}.$$

Letting $\varepsilon \downarrow 0$, we obtain $v(0, S(0); \alpha) \geq \mathbb{E}\big[e^{-rT}\widehat{\varphi}_\alpha(S(T))\big]$. This proves (2.7).

6 Examples

In this section, we give examples of options whose upper hedging prices can be computed using either Theorem 3.1 or 5.3. In both these theorems, the path-dependent payoff function g is assumed to be lower semicontinuous. Some option contracts are written with upper-semicontinuous payoffs. However, one can usually trivially modify an upper-semicontinuous payoff to obtain a lower-semicontinuous payoff, and then our theorems apply. Our first example highlights the danger of applying them naively to upper-semicontinuous payoffs.

Example 6.1 (Cactus option) Consider an option whose payoff at expiration date T is 1 if and only if $S(T) = K$, where K is a fixed positive number.

Otherwise, the payoff is zero. The payoff can be written as $\varphi(S(T))$, where $\varphi(x) \triangleq I_{\{x=K\}}$ is upper semicontinuous rather than lower semicontinuous. If we ignore this fact and attempt to use Theorem 2.2 (which via Remark 5.4 follows from Theorem 5.3) to compute the upper hedging price, we would first determine

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda \geq 0} e^{-\alpha\lambda}\varphi\big(xe^{-\lambda}\big) = \left(\frac{K}{x}\right)^\alpha I_{\{x \geq K\}}, \quad x \geq 0,$$

and then compute $\mathbb{E}\big[e^{-rT}\widehat{\varphi}_\alpha(S(T))\big]$. This last quantity is strictly positive. However, the option is clearly worth zero, since there is zero probability that $S(T) = K$. To correctly compute the upper hedging price, one should replace the given φ by its lower-semicontinuous envelope $\varphi_* \equiv 0$.

Example 6.2 (Digital put) The payoff for the digital put is

$$g(S(\cdot)) = I_{\{S(T)<K\}}$$

where K is positive. This can be written as $g(S(\cdot)) = \varphi(S(T))$, where $\varphi(x) \triangleq I_{\{x<K\}}$. According to Theorem 2.2, we should first determine the face-lift

$$\widehat{\varphi}_\alpha(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq K, \\ (K/x)^\alpha & \text{if } x \geq K, \end{cases}$$

and then the upper hedging price can be computed to be

$$\begin{aligned} v(0, S(0); \alpha) &= e^{-rT}\,\mathbb{E}\big[\widehat{\varphi}_\alpha(S(T))\big] \\ &= e^{-rT}N(-d) + e^{(1+\alpha)(\frac{1}{2}\alpha\sigma^2 - r)T}\left(\frac{K}{S(0)}\right)^\alpha N\big(d - \alpha\sigma\sqrt{T}\big), \end{aligned}$$

where $d \triangleq \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]$.

Example 6.3 (Discrete barrier option) The in-the-money knock-out call described in Section 1 was discussed in considerable detail in Section 4. Here we modify the payoff by assuming the option can only knock out at discrete check times $0 < t_1 < t_2 < \cdots < t_I \leq T$, i.e.,

$$g(S(\cdot)) = \big(S(T) - K\big)^+ \prod_{i=1}^{I} I_{\{S(t_i)<B\}}.$$

The payoff function g is of the form treated in Proposition 5.2, and thus

$$g_*(S, \lambda) = \big(S(T)e^{-\lambda(T)} - K\big)^+ \prod_{i=1}^{I} I_{\{S(t_i)e^{-\lambda(t_i)}<B\}}.$$

The supremum in (5.7) is approached by processes λ which are constant between the check times, and jump at the check times "just enough" to prevent knock-out. More precisely, let $\{B_n\}_{n=1}^\infty$ be converging up to B. For each n, define

$$\lambda_n(t) \triangleq \max_{\{i;\, t_i \leq t\}} \big(\log S(t_i) - \log B_n\big)^+, \quad 0 \leq t \leq T. \tag{6.1}$$

Then $S(t_i)e^{-\lambda_n(t_i)} \le B_n$ for each $i \in \{1,\ldots,I\}$, and λ_n is the smallest process in $\mathcal{R}$ which forces these inequalities. Starting with the maximization problem (5.7) in Theorem 5.3 and applying the arguments which led to (4.6) and (4.7), we obtain for the upper hedging price

$$\begin{aligned} v(0,S(0);\alpha) &= \lim_{n\to\infty} \mathbb{E}\big[e^{-rT-\alpha\lambda_n(T)}\big(S(T)e^{-\lambda_n(T)} - K\big)^+\big] \\ &= \mathbb{E}\big[e^{-rT-\alpha\lambda^*(T)}\big(S(T)e^{-\lambda^*(T)} - K\big)^+\big], \end{aligned}$$

where λ^* is given by (6.1) with B in place of B_n. This may be rewritten as

$$v(0,S(0);\alpha) = e^{-rT}\,\mathbb{E}\left[\left(1 \wedge \min_{1\le i\le I} \frac{B}{S(t_i)}\right)^\alpha \left(S(T)\left(1 \wedge \min_{1\le i\le I} \frac{B}{S(t_i)}\right) - K\right)^+\right].$$

The computation has been reduced to a finite-dimensional Gaussian integration. If the barrier depends on time, we need only to replace the ratios $B/S(t_i)$ by $B(t_i)/S(t_i)$ in the last formula.

Example 6.4 (Vanilla put) We compute the upper hedging price of the vanilla put as a prelude to Examples 6.5 and 6.6. The payoff of the vanilla put is $g(S(\cdot)) = \varphi(S(T))$, where $\varphi(x) = (K-x)^+$ and K is a positive constant. According to Proposition 5.2 and Theorem 5.3, the upper hedging price is

$$v(0,S(0);\alpha) = \sup_{\lambda\in\mathcal{R}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(K - S(T)e^{-\lambda(T)}\big)^+\big], \tag{6.2}$$

where we take $I=1$ and $t_1 = T$ in the definition of $\mathcal{R}$, meaning that the processes are continuous except for a possible jump at time T. Theorem 2.2 applies, and asserts that $v(0,S(0);\alpha) = e^{-rT}\,\mathbb{E}\big[\widehat{\varphi}_\alpha(S(T);K)\big]$, where the face-lift is given by

$$\widehat{\varphi}_\alpha(x;K) \triangleq \sup_{\lambda\ge 0} e^{-\alpha\lambda}\big(K - xe^{-\lambda}\big)^+ = \begin{cases} K - x & \text{if } 0 \le x \le \frac{\alpha K}{1+\alpha}, \\ \frac{K}{1+\alpha}\big(\frac{\alpha K}{(1+\alpha)x}\big)^\alpha & \text{if } x \ge \frac{\alpha K}{1+\alpha}. \end{cases} \tag{6.3}$$

On the other hand, in the case $\alpha > 0$, maximizing the integrand in (6.2) for every value of $S(T)$ shows that a process $\lambda \in \mathcal{R}$ is a maximizer if

$$\lambda(T) = \left(\log S(T) - \log \frac{\alpha K}{1+\alpha}\right)^+.$$

Example 6.5 (Lookback put) We consider the lookback put payoff function $g(S(\cdot)) = M(S) - S(T)$ with maximum $M(S) \triangleq \sup_{0\le t\le T} S(t)$. According to Proposition 5.2 and Theorem 5.3, the upper hedging price is

$$v(0,S(0);\alpha) = \sup_{\lambda\in\mathcal{R}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(M\big(Se^{-\lambda}\big) - S(T)e^{-\lambda(T)}\big)\big].$$

We maximize $M(Se^{-\lambda})$ over λ by choosing λ to be identically zero on $[0,T)$, and this results in $M(Se^{-\lambda}) = M(S)$. The upper hedging price is obtained by then choosing $\lambda(T) \ge 0$ so as to maximize $\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(M(S) - S(T)e^{-\lambda(T)}\big)\big]$.

This is the maximization problem of (6.2) with $M(S)$ replacing the strike price K. While Theorem 2.2 does not apply, direct calculation as in Example 6.4 for $\alpha > 0$ shows that a maximizing process in $\mathcal{R}$ is given by

$$\lambda^*(t) = \left(\log S(T) - \log \frac{\alpha M(S)}{1+\alpha}\right)^+ I_{\{t=T\}}, \quad 0 \le t \le T.$$

Example 6.6 (Asian put) We next consider the Asian payoff function given by $g(S(\cdot)) = \big(A(S) - S(T)\big)^+$ with arithmetic average $A(S) \triangleq \frac{1}{T}\int_0^T S(t)\,dt$. The upper hedging price is

$$v(0, S(0); \alpha) = \sup_{\lambda \in \mathcal{R}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(A(Se^{-\lambda}) - S(T)e^{-\lambda(T)}\big)^+\big].$$

Once again, for $\alpha > 0$, a maximizing λ is identically zero on $[0, T)$, and for this process $A(Se^{-\lambda}) = A(S)$. The upper hedging price is obtained by choosing $\lambda(T) \ge 0$ so as to maximize $\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(A(S) - S(T)e^{-\lambda(T)}\big)^+\big]$. This is the maximization problem of (6.2) with $A(S)$ replacing the strike price K. A maximizing process in $\mathcal{R}$ is

$$\lambda^*(t) = \left(\log S(T) - \log \frac{\alpha A(S)}{1+\alpha}\right)^+ I_{\{t=T\}}, \quad t \in [0, T].$$

Example 6.7 (Book of two barrier options) Upper hedging prices for individual exotic options are often too high to permit sales except in thinly traded over-the-counter markets. However, the theory developed in this paper can be applied to books of derivatives, and because the upper hedging methodology exploits natural hedges within the book, the upper hedging price of the book can be considerably less than the sum of the upper hedging prices of the individual assets in the book. The difficulty with pricing books, of course, is in solving the resulting stochastic control problem of Theorem 3.1 or 5.3.

In this example, we determine the upper hedging price of a book of two in-the-money knock-out calls of the type discussed in Sections 1 and 4. These calls have a common maturity T, a common strike price K, and the barriers L and U are related by $0 < K < L < U$. The "low barrier" call has payoff

$$g^L(S(\cdot)) = \big(S(T) - K\big)^+ I_{\{\max_{0 \le t \le T} S(t) < L\}},$$

and the "high barrier" call has corresponing payoff $g^U(S(\cdot))$. Let $v^L(t, x; \alpha)$ and $v^U(t, x; \alpha)$ be the functions given by (4.8), (4.10) with B replaced by L and U, respectively. These functions provide the upper hedging prices of the calls, except at the respective barriers. They further satisfy their respective versions of (4.13), (4.14) and (4.15).

Using Theorem 3.1 to determine the upper hedging price of a book consisting of one call with barrier U and $\kappa \ge 0$ calls with barrier L, we must compute

$$v(0, S(0); \alpha) = \sup_{\lambda \in \mathcal{C}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(\kappa g^L(Se^{-\lambda}) + g^U(Se^{-\lambda})\big)\big]. \qquad (6.4)$$

Using notation from (4.4), we shall instead compute

$$v^*(0,S(0);\alpha) = \sup_{\lambda\in\mathcal{C}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}(S_\lambda(T)-K)^+\big(\kappa I_{\{M_\lambda(T)\le L\}} + I_{\{M_\lambda(T)\le U\}}\big)\big] \quad (6.5)$$

and subsequently argue that $v(0,S(0);\alpha)$ and $v^*(0,S(0);\alpha)$ agree except when $S(0)=L$ or $S(0)=U$, in which cases we have $v(0,L;\alpha)=\lim_{x\downarrow L} v^*(0,x;\alpha)$ and $v(0,U;\alpha)=0$.

We first construct a function $w(t,x,y)$ which we shall show is almost the upper hedging price of the book at time $t\in[0,T]$ if at that time the stock price is $S(t)=x>0$ and the maximum stock price to date is $M(t)\triangleq \max_{0\le u\le t} S(u) = y\ge x$. We begin by setting

$$w(t,x,y)\triangleq v^U(t,x;\alpha)I_{\{y\le U\}},\quad 0\le t\le T,\ y>L,\ 0<x\le y. \quad (6.6)$$

We next define $\tilde{\kappa}\triangleq\kappa+1$ and the deterministic time

$$t^*\triangleq T\wedge\min\big\{t\ge 0;\ \tilde{\kappa}v^L(s,L;\alpha)\ge v^U(s,L;\alpha)\ \forall\, s\in[t,T]\big\}.$$

We set

$$w(t,x,y)\triangleq\tilde{\kappa}v^L(t,x;\alpha),\quad t^*<t\le T,\ 0<x\le y\le L. \quad (6.7)$$

Finally, for $0\le t\le t^*$ and $0\le x\le y\le L$, we define $w(t,x,y)$ to be the solution to the Black-Scholes partial differential equation

$$w_t(t,x)+rxw_x(t,x)+\frac{1}{2}\sigma^2x^2w_{xx}(t,x)=rw(t,x), \quad (6.8)$$

subject to the boundary conditions

$$w(t,0)=0,\qquad 0\le t<t^*, \quad (6.9)$$

$$w(t,L)=v^U(t,L;\alpha),\qquad 0\le t<t^*, \quad (6.10)$$

$$w(t^*,x)=\tilde{\kappa}v^L(t^*,x;\alpha),\quad 0<x\le L. \quad (6.11)$$

In other words, in the region $0\le t\le t^*$, $0<x\le y\le L$, the function $w(t,x,y)$ is the Black-Scholes price of a derivative security which knocks out at L, paying rebate $v^U(s,L;\alpha)$ if the knock-out occurs at time $s<t^*$, and otherwise expires at time t^*, paying $\tilde{\kappa}v^L(t^*,S(t^*);\alpha)$ upon expiration. Because $\alpha w(t,x)+xw_x(t,x)$ also satisfies the Black-Scholes equation and $\alpha w(t,x)+xw_x(t,x)\ge 0$ on the boundary $\big(\{t^*\}\times[0,L)\big)\cup\big([0,t^*)\times\{L\}\big)$ (see (4.15), satisfied by both v^L and v^U), we have from the maximum principle that

$$\alpha w(t,x)+xw_x(t,x)\ge 0,\quad 0\le t\le t^*,\ 0\le x\le L. \quad (6.12)$$

We show that $v^*(0,S(0);\alpha)$ given by (6.5) is actually $w(0,S(0),S(0))$. If $S(0)>U$, then both $v^*(0,S(0);\alpha)$ and $w(0,S(0),S(0))$ are zero. If $L<S(0)\le U$, then the computation of $v^*(t,S(0);\alpha)$ reduces to the single-option problem

solved in Corollary 4.1 with $B = U$, and hence $v^*(0, S(0); \alpha) = v^U(0, S(0); \alpha) = w(t, S(0), S(0))$ by (6.6).

In remains to prove the equality when $0 < S(0) \leq L$. This requires the proof of an inequality in each direction. For the first inequality, we let $\lambda \in \mathcal{C}$ be given and define $\Theta_L \triangleq \inf\{t \geq 0;\ S_\lambda(t) > L\}$, $\Theta_U \triangleq \inf\{t \geq 0;\ S_\lambda(t) > U\}$. Itô's formula implies

$$\begin{aligned}
&w(0, S(0), S(0)) \\
&= w(0, S(0)) \qquad\qquad (6.13)\\
&= \mathbb{E}\big[e^{-r(t^*\wedge\Theta_L)-\alpha\lambda(t^*\wedge\Theta_L)}\, w(t^*\wedge\Theta_L, S_\lambda(t^*\wedge\Theta_L))\big] \\
&\quad + \mathbb{E}\bigg[\int_0^{t^*\wedge\Theta_L} e^{-rt-\alpha\lambda(t)}\big(\alpha w(t, S_\lambda(t)) + S_\lambda(t) w(t, S_\lambda(t))\big)\, d\lambda(t)\bigg] \\
&\geq \mathbb{E}\big[e^{-r\Theta_L-\alpha\lambda(\Theta_L)}\, v^U(\Theta_L, L; \alpha)\, I_{\{\Theta_L < t^*\}} \\
&\qquad + \tilde{\kappa} e^{-rt^*-\alpha\lambda(t^*)}\, v^L(t^*, S_\lambda(t^*); \alpha)\, I_{\{\Theta_L \geq t^*\}}\big].
\end{aligned}$$

We continue with the case $\Theta_L < t^*$, again using Itô's formula, this time to obtain

$$\begin{aligned}
&\mathbb{E}\big[e^{-r\Theta_L-\alpha\lambda(\Theta_L)}\, v^U(\Theta_L, L; \alpha)\, I_{\{\Theta_L < t^*\}}\big] \\
&= \mathbb{E}\big[e^{-r(T\wedge\Theta_U)-\alpha\lambda(T\wedge\Theta_U)}\, v^U\big(T\wedge\Theta_U, S_\lambda(T\wedge\Theta_U); \alpha\big)\, I_{\{\Theta_L < t^*\}}\big] \\
&\quad + \mathbb{E}\bigg[\int_{\Theta_L}^{T\wedge\Theta_U} e^{-rt-\alpha\lambda(t)}\big(\alpha v^U(t, S_\lambda(t); \alpha) \qquad\qquad (6.14)\\
&\qquad\qquad + S_\lambda(t)\, v_x^U(t, S_\lambda(t); \alpha)\big)\, d\lambda(t)\, I_{\{\Theta_L < t^*\}}\bigg] \\
&\geq \mathbb{E}\big[e^{-r\Theta_U-\alpha\lambda(\Theta_U)}\, v^U(\Theta_U, U; \alpha)\, I_{\{\Theta_L < t^*, \Theta_U < T\}} \\
&\qquad\qquad + e^{-rT-\alpha\lambda(T)}\, v^U(T, S_\lambda(T); \alpha)\, I_{\{\Theta_L < t^*, \Theta_U \geq T\}}\big] \\
&\geq \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(S_\lambda(T) - K\big)^+ I_{\{\Theta_L < t^*, M_\lambda(T) \leq U\}}\big].
\end{aligned}$$

We also continue (6.13) in the case $\Theta_L \geq t^*$. In this case, we have

$$\begin{aligned}
&\tilde{\kappa}\, \mathbb{E}\big[e^{-rt^*-\alpha\lambda(t^*)}\, v^L(t^*, S_\lambda(t^*); \alpha)\, I_{\{\Theta_L \geq t^*\}}\big] \\
&= \tilde{\kappa}\, \mathbb{E}\big[e^{-r(T\wedge\Theta_L)-\alpha\lambda(T\wedge\Theta_L)}\, v^L\big(T\wedge\Theta_L, S_\lambda(T\wedge\Theta_L); \alpha\big)\, I_{\{\Theta_L \geq t^*\}}\big] \\
&\quad + \tilde{\kappa}\, \mathbb{E}\bigg[\int_{t^*}^{T\wedge\Theta_L} e^{-rt-\alpha\lambda(t)}\big(\alpha v^L(t, S_\lambda(t); \alpha) \qquad\qquad (6.15)\\
&\qquad\qquad + S_\lambda(t)\, v_x^L(t, S_\lambda(t); \alpha)\big)\, d\lambda(t)\, I_{\{\Theta_L \geq t^*\}}\bigg] \\
&\geq \tilde{\kappa}\, \mathbb{E}\big[e^{-r\Theta_L-\alpha\lambda(\Theta_L)}\, v^L(\Theta_L, L; \alpha)\, I_{\{t^* \leq \Theta_L < T\}}\big] \\
&\quad + \tilde{\kappa}\, \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\, v^L(T, S_\lambda(T); \alpha)\, I_{\{\Theta_L \geq T\}}\big] \\
&\geq \mathbb{E}\big[e^{-r\Theta_L-\alpha\lambda(\Theta_L)}\, v^U(\Theta_L, L; \alpha)\, I_{\{t^* \leq \Theta_L < T\}}\big] \\
&\quad + \tilde{\kappa}\, \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(S_\lambda(T) - K\big)^+ I_{\{M_\lambda(T) \leq L\}}\big],
\end{aligned}$$

where the definition of t^* is used to obtain the last inequality. Finally, Itô's formula implies

$$
\begin{aligned}
&\mathbb{E}\big[e^{-r\Theta_L-\alpha\lambda(\Theta_L)}\,v^U(\Theta_L,L;\alpha)\,I_{\{t^*\le\Theta_L<T\}}\big]\\
&=\mathbb{E}\big[e^{-r(T\wedge\Theta_U)-\alpha\lambda(T\wedge\Theta_U)}\,v^U(T\wedge\Theta_U,S_\lambda(T\wedge\Theta_U);\alpha)\,I_{\{t^*\le\Theta_L<T\}}\big]\\
&\quad+\mathbb{E}\bigg[\int_{\Theta_L}^{T\wedge\Theta_U} e^{-rt-\alpha\lambda(t)}\big(\alpha v^U(t,S_\lambda(t);\alpha) \qquad (6.16)\\
&\qquad\qquad+S_\lambda(t)\,v_x^U(t,S_\lambda(t);\alpha)\big)\,d\lambda(t)\,I_{\{t^*\le\Theta_L<T\}}\bigg]\\
&\ge\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\,v^U(T,S_\lambda(T);\alpha)\,I_{\{t^*\le\Theta_L<T,\,\Theta_U\ge T\}}\big]\\
&\ge\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(S_\lambda(T)-K\big)^+\,I_{\{t^*\le\Theta_L<T,\,M_\lambda(T)\le U\}}\big].
\end{aligned}
$$

Combining (6.13)–(6.16), we see that

$$
\begin{aligned}
&w(0,S(0),S(0)) \qquad (6.17)\\
&\ge\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(S_\lambda(T)-K\big)^+\big(I_{\{\Theta_L<T,\,M_\lambda(T)\le U\}}+\tilde{\kappa}I_{\{M_\lambda(T)\le L\}}\big)\big]\\
&=\mathbb{E}\big[e^{-rT-\alpha\lambda(T)}\big(S_\lambda(T)-K\big)^+\big(\kappa I_{\{M_\lambda(T)\le L\}}+I_{\{M_\lambda(T)\le U\}}\big)\big].
\end{aligned}
$$

Recalling (6.5) and using the fact that $\lambda\in\mathcal{C}$ is arbitrary, we conclude

$$
w(0,S(0),S(0))\ge v^*(0,S(0);\alpha). \qquad (6.18)
$$

For the reverse inequality in the case $0<S(0)\le L$, we define

$$
\lambda^*(t)=\begin{cases}\max\limits_{0\le u\le t}\big(\log S(u)-\log U\big)^+ & \text{if } 0\le t\le t^*,\\[1ex] \max\limits_{0\le u\le t}\big(\log S(u)-\log U\big)^+ I_{\{M(t^*)>L\}} & \\ \quad+\max\limits_{t^*\le u\le t}\big(\log S(u)-\log L\big)^+ I_{\{M(t^*)\le L\}} & \text{if } t^*\le t\le T.\end{cases} \qquad (6.19)
$$

Then S_{λ^*} never exceeds U, and if by time t^*, S_{λ^*} has not exceeded L, then it never exceeds L. The process λ^* is the minimal process which guarantees these properties; in particular, λ^* grows only when S_{λ^*} is at either U or L. Replacing λ in (6.13) by λ^*, we have equality because $\lambda^*\equiv 0$ on $[0,t^*\wedge\Theta_L]$. Replacing λ by λ^* in (6.14), we again have equality because on the set $\{\Theta_L<t^*\}$, the process λ^* grows only when $S_{\lambda^*}=U$, and

$$
\alpha v^U(t,U;\alpha)+U\,v_x^U(t,U;\alpha)=0.
$$

Furthermore, $\{\Theta_L<t^*,\Theta_U<T\}=\varnothing$, since $M_{\lambda^*}(T)\le U$. With λ replaced by λ^*, (6.15) becomes an equality because on the set $t^*\le t\le T\wedge\Theta_L$, the process λ^* grows only when $S_{\lambda^*}=L$ and

$$
\alpha v^L(t,L;\alpha)+Lv_x^L(t,L;\alpha)=0.
$$

Furthermore, $\{t^* \le \Theta_L < T\} = \varnothing$ because $M_{\lambda^*}(T) \le L$ on $\Theta_L(T) \ge t^*$. For this same reason, all terms in (6.16) are zero when λ is replaced by λ^*. This leads to equality in (6.17) when λ is replaced by λ^*, and according to (6.5).

$$\begin{aligned} w(0,S(0),S(0)) &= \mathbb{E}\big[e^{-rT-\alpha\lambda^*(T)}(S_{\lambda^*}(T)-K)^+\big(\kappa I_{\{M_{\lambda^*}(T)\le L\}} + I_{\{M_{\lambda^*}(T)\le U\}}\big)\big] \\ &\le v^*(0,S(0);\alpha). \end{aligned} \tag{6.20}$$

Due to (6.18), equality has to hold here.

To establish the relationship between $v^*(0,S(0);\alpha)$ and the upper hedging price $v(0,S(0);\alpha)$ of (6.4), we start with the case $S(0) < U$ and choose two sequences of barriers $\{L_n\}_{n=1}^\infty$ and $\{U_n\}_{n=1}^\infty$ with $L_n \uparrow L$ and $U_n \uparrow U$ satisfying $L_n < L \le U_n$ and $S(0) \le U_n < U$ for all $n \in \mathbb{N}$. Let λ_n be given by (6.19) with L and U replaced by L_n and U_n, respectively. Then $\lambda_n \in \mathcal{C}$ and $\lambda_n \downarrow \lambda^*$ pointwise, so $S_{\lambda_n} \uparrow S_{\lambda^*}$ and $M_{\lambda_n} \uparrow M_{\lambda^*}$ pointwise. In addition, $\{M_{\lambda_n}(T) \le U_n\} = \Omega = \{M_{\lambda^*}(T) \le U\}$ and for $S(0) \in (0,U) \setminus \{L\}$, we also have

$$\lim_{n\to\infty} I_{\{M_{\lambda_n}(T)\le L_n\}} = \lim_{n\to\infty} I_{\{M(t^*)\le L_n\}} = I_{\{M(t^*)\le L\}} = I_{\{M_{\lambda^*}(T)\le L\}}, \quad \text{a.s.}$$

It follows from (6.4), $L_n < L$, $U_n < U$, Fatou's lemma and (6.20) with equality that, for $S(0) \in (0,U) \setminus \{L\}$,

$$\begin{aligned} &v(0,S(0);\alpha) \\ &\quad \ge \liminf_{n\to\infty} \mathbb{E}\big[e^{-rT-\lambda_n(T)}\big(S_{\lambda_n}(T)-K\big)^+\big(\kappa I_{\{M_{\lambda_n}(T)<L\}} + I_{\{M_{\lambda_n}(T)<U\}}\big)\big] \\ &\quad \ge \liminf_{n\to\infty} \mathbb{E}\big[e^{-rT-\lambda_n(T)}\big(S_{\lambda_n}(T)-K\big)^+\big(\kappa I_{\{M_{\lambda_n}(T)\le L_n\}} + I_{\{M_{\lambda_n}(T)\le U_n\}}\big)\big] \\ &\quad \ge \mathbb{E}\big[e^{-rT-\lambda^*(T)}\big(S_{\lambda^*}(T)-K\big)^+\big(\kappa I_{\{M_{\lambda^*}(T)\le L\}} + I_{\{M_{\lambda^*}(T)\le U\}}\big)\big] \\ &\quad = v^*(0,S(0);\alpha). \end{aligned}$$

The reverse inequality is obvious. Since the case $S(0) > U$ is trivial, we have established

$$v(0,S(0);\alpha) = v^*(0,S(0);\alpha), \quad \forall\, S(0) \in (0,\infty) \setminus \{L,U\}.$$

It is clear that $v(0,U;\alpha) = 0$, since both options are knocked-out at the initial time. Finally, if $S(0) = L$, then the "low barrier" option is knocked out at the initial time, and by Corollary 4.1,

$$v(0,L;\alpha) = v^U(0,L;\alpha) = \lim_{x\downarrow L} v^U(0,x;\alpha) = \lim_{x\downarrow L} v^*(0,x;\alpha).$$

The construction of the upper hedging price for the book of two barrier options is complete. $\diamondsuit$

7 Proof of Theorem 3.1

We denote by

$$\begin{aligned} \mathcal{L} \triangleq \big\{\, \lambda;\ &\lambda \text{ is a nondecreasing, } \{\mathcal{F}(t); 0 \le t \le T\}\text{-adapted process,} \\ &\text{Lipschitz in } t \text{ uniformly in } \omega, \text{ with } \lambda(0) = 0 \,\big\}, \end{aligned}$$

the class of processes over which the supremum in (2.4) is taken. In a first step, we show that the supremum in (2.4) can be reduced to the set $\mathcal{L}_{\rm pl}$ of piecewise linear, $\{\mathcal{F}^W(t); 0 \le t \le T\}$-adapted processes $\lambda \in \mathcal{L}$, meaning that for every $\lambda \in \mathcal{L}_{\rm pl}$ there exist $m \in \mathbb{N}$, a partition $0 = t_0 < t_1 < \cdots < t_m = T$, and bounded, $\mathcal{F}^W(t_i)$-measurable $a_i \colon \Omega \to [0,\infty)$ such that

$$\lambda(t,\omega) = \sum_{i=0}^{m-1} a_i(\omega)\big((t_{i+1} \wedge t) - t_i\big)^+, \quad t \in [0,T],\, \omega \in \Omega. \tag{7.1}$$

Lemma 7.1 *Let $g : C_+[0,T] \to [0,\infty)$ be a measurable function. We have*

$$\sup_{\lambda \in \mathcal{L}} \mathbb{E}_\lambda\big[e^{-rT-\alpha\lambda(T)} g(S)\big] = \sup_{\lambda \in \mathcal{L}_{\rm pl}} \mathbb{E}_\lambda\big[e^{-rT-\alpha\lambda(T)} g(S)\big].$$

Proof: Since $\mathcal{L}_{\rm pl} \subset \mathcal{L}$, there is just one inequality to prove. Consider $\lambda \in \mathcal{L}$. Then there is a bounded, adapted process $\lambda' \colon [0,T] \times \Omega \to [0,\infty)$ such that

$$\lambda(t,\omega) = \int_0^t \lambda'(s,\omega)\, ds, \quad t \in [0,T].$$

As in the construction of the stochastic integral (see, e.g., [12], Chap. 3, Lemma 2.4), one can prove the existence of a sequence $\{\lambda'_n\}_{n\in\mathbb{N}}$ of processes $\lambda'_n \colon [0,T] \times \Omega \to [0,\infty)$ of the form

$$\lambda'_n(t,\omega) = \sum_{i=0}^{m_n-1} a_{i,n}(\omega) 1_{(t_{i,n}, t_{i+1,n}]}(t), \quad t \in [0,T],\, \omega \in \Omega,$$

where $m_n \in \mathbb{N}$, $0 = t_{0,n} < t_{1,n} < \cdots < t_{m_n,n} = T$ and every $a_{i,n} \colon \Omega \to [0,\infty)$ is $\mathcal{F}(t_{i,n})$-measurable and bounded by the Lipschitz constant of λ, such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |\lambda'_n(t) - \lambda'(t)|^2\, dt\right] = 0.$$

By changing $a_{i,n}$ on a set of $\mathbb{P}$-measure zero if necessary, we may assume that every $a_{i,n}$ is $\mathcal{F}^W(t_{i,n})$-measurable. By the definition of the stochastic integral,

$$\lim_{n\to\infty} \int_0^T \lambda'_n(t)\, dW(t) = \int_0^T \lambda'(t)\, dW(t) \tag{7.2}$$

in $L^2(\Omega, \mathcal{F}(T), \mathbb{P})$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} \int_0^T |\lambda'_n(t) - \lambda'(t)|^2\, dt = 0$ $\mathbb{P}$-almost surely and that the convergence in (7.2) is also $\mathbb{P}$-almost sure. Define

$$\lambda_n(t,\omega) = \int_0^t \lambda'_n(s,\omega)\, ds, \quad t \in [0,T],\, \omega \in \Omega.$$

Then $\lambda_n(T) \to \lambda(T)$ almost surely. Let Z_λ denote the density given by (2.5) and let Z_{λ_n} denote the corresponding density for λ_n. By Fatou's lemma,

$$\begin{aligned}\mathbb{E}_\lambda[e^{-\alpha\lambda(T)} f] &= \mathbb{E}[e^{-\alpha\lambda(T)} f Z_\lambda] \\ &\le \liminf_{n\to\infty} \mathbb{E}[e^{-\alpha\lambda_n(T)} f Z_{\lambda_n}] = \liminf_{n\to\infty} \mathbb{E}_{\lambda_n}[e^{-\alpha\lambda_n(T)} f],\end{aligned}$$

where $f \triangleq e^{-rT} g(S)$. ◇

Lemma 7.2 *Let $g : C_+[0,T] \to [0,\infty)$ be a lower-semicontinuous function. We have*

$$\sup_{\lambda\in\mathcal{L}_{\mathrm{pl}}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)} g\big(Se^{-\lambda}\big)\big] = \sup_{\lambda\in\mathcal{C}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)} g\big(Se^{-\lambda}\big)\big].$$

Proof: Since $\mathcal{L}_{\mathrm{pl}} \subset \mathcal{C}$, there is just one inequality to prove. Consider $\lambda \in \mathcal{C}$. Given $n \in \mathbb{N}$, define $a_{0,n} = 0$, $t_{0,n} = 0$, $t_{i,n} = iT2^{-n}$ and

$$a_{i,n}(\omega) = \min\Big\{2^n, \frac{\lambda(t_{i,n},\omega) - \lambda(t_{i-1,n},\omega)}{T2^{-n}}\Big\} \tag{7.3}$$

for all $i \in \{1, 2, \ldots, 2^n\}$ and $\omega \in \Omega$. By changing $a_{i,n}$ on a set of $\mathbb{P}$-measure zero if necessary, we may assume that $a_{i,n}$ is $\mathcal{F}^W(t_{i,n})$-measurable for every $i \in \{1, \ldots, 2^n\}$. Define $\lambda_n \in \mathcal{L}_{\mathrm{pl}}$ by

$$\lambda_n(t,\omega) = \sum_{i=0}^{2^n-1} a_{i,n}(\omega)\big((t_{i+1,n} \wedge t) - t_{i,n}\big)^+, \quad t \in [0,T],\ \omega \in \Omega.$$

To prove the $\mathbb{P}$-almost sure uniform convergence of $\{\lambda_n\}_{n\in\mathbb{N}}$ to λ, consider an $\omega \in \Omega$ so that (7.3) holds for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, 2^n\}$. Since $t \mapsto \lambda(t,\omega)$ is uniformly continuous and nondecreasing on $[0,T]$, the modulus of continuity

$$m(t,\omega) = \sup_{s\in[0,T-t]} (\lambda(s+t,\omega) - \lambda(s,\omega)), \quad t \in [0,T],$$

is a nondecreasing continuous function with $m(0,\omega) = 0$. In particular, there exists $k \in \mathbb{N}$ satisfying $m(T2^{-k},\omega) < T$. For every $n \geq k$ and $i \in \{1, \ldots, 2^n\}$, the quotient in (7.3) is less than 2^n; hence $\lambda_n(t_{i+1,n},\omega) = \lambda(t_{i,n},\omega)$ for all $i \in \{0, \ldots, 2^n-1\}$. If $t \in [t_{i,n}, t_{i+1,n}]$ with $i \in \{1, \ldots, 2^n-1\}$, then $\lambda(t_{i-1,n},\omega) = \lambda_n(t_{i,n},\omega) \leq \lambda_n(t,\omega) \leq \lambda_n(t_{i+1,n},\omega) = \lambda(t_{i,n},\omega)$, because λ_n is nondecreasing. Note that $\lambda_n(t,\omega) = 0$ for $t \in [0, t_{1,n}]$. Because λ is also nondecreasing, we have

$$\sup_{t\in[0,T]} |\lambda(t,\omega) - \lambda_n(t,\omega)| \leq m(T2^{-n},\omega) \downarrow 0 \quad \text{as } n \to \infty.$$

Using the lower semicontinuity of g and Fatou's lemma, we compute

$$\begin{aligned}\mathbb{E}\big[e^{-rT-\alpha\lambda(T)} g\big(Se^{-\lambda}\big)\big] &\leq \mathbb{E}\Big[e^{-rT-\alpha\lambda(T)} \liminf_{n\to\infty} g\big(Se^{-\lambda_n}\big)\Big] \\ &\leq \liminf_{n\to\infty} \mathbb{E}\big[e^{-rT-\alpha\lambda_n(T)} g\big(Se^{-\lambda_n}\big)\big],\end{aligned}$$

which implies the desired inequality. $\diamondsuit$

Proof of Theorem 3.1: By Theorem 2.1 and Lemmas 7.1, 7.2, it suffices to show

$$\sup_{\lambda\in\mathcal{L}_{\mathrm{pl}}} \mathbb{E}_\lambda\big[e^{-rT-\alpha\lambda(T)} g(S)\big] = \sup_{\bar\lambda\in\mathcal{L}_{\mathrm{pl}}} \mathbb{E}\big[e^{-rT-\alpha\bar\lambda(T)} g\big(Se^{-\bar\lambda}\big)\big].$$

In other words, for each $\lambda \in \mathcal{L}_{\mathrm{pl}}$, we will construct a $\bar\lambda \in \mathcal{L}_{\mathrm{pl}}$ such that

$$\mathbb{E}_\lambda\big[e^{-rT-\alpha\lambda(T)} g(S)\big] = \mathbb{E}\big[e^{-rT-\alpha\bar\lambda(T)} g\big(Se^{-\bar\lambda}\big)\big], \tag{7.4}$$

and conversely, for each $\bar{\lambda} \in \mathcal{L}_{\mathrm{pl}}$, we will construct $\lambda \in \mathcal{L}_{\mathrm{pl}}$ satisfying (7.4).

For each $\lambda \in \mathcal{L}_{\mathrm{pl}}$, we define $\varphi_\lambda \colon \Omega \to \Omega$ by

$$\varphi_\lambda(\omega)(t) \triangleq \omega(t) + \frac{1}{\sigma}\lambda(t,\omega), \quad t \in [0,T],\ \omega \in \Omega. \tag{7.5}$$

Note that φ_λ is $\mathcal{F}^W(t)/\mathcal{F}^W(t)$-measurable for every $t \in [0,T]$. We show that φ_λ is bijective. To see that φ_λ is injective, we suppose that $\varphi_\lambda(\omega_1) = \varphi_\lambda(\omega_2)$. The process λ has a representation of the form (7.1), and in terms of this representation we define $I \triangleq \max\{i;\, \omega_1(t) = \omega_2(t)\ \forall\, t \in [0,t_i]\}$. Note that $0 \le I \le m$. If $I < m$, then $a_i(\omega_1) = a_i(\omega_2)$ for all $i \le I$, which implies that $\lambda(t,\omega_1) = \lambda(t,\omega_2)$ for all $t \le t_{I+1}$. Therefore, for $0 \le t \le t_{I+1}$, we have

$$\omega_1(t) = \varphi_\lambda(\omega_1)(t) - \frac{1}{\sigma}\lambda(t,\omega_1) = \varphi_\lambda(\omega_2)(t) - \frac{1}{\sigma}\lambda(t,\omega_2) = \omega_2(t),$$

and the definition of I is contradicted. It follows that $I = m$ and $\omega_1 = \omega_2$. To see that φ_λ is surjective, we let $\varpi \in \Omega$ be given. We set $\omega(0) = 0$ and define inductively, for $i \in \{0,1,\dots,m-1\}$,

$$\omega(t) \triangleq \varpi(t) - \frac{1}{\sigma}\sum_{j=0}^{i} a_j(\omega)\big((t_{j+1}\wedge t) - t_j\big)^+, \quad t_i < t \le t_{i+1}.$$

Then $\varpi = \varphi_\lambda(\omega)$. This construction also shows that φ_λ^{-1} is $\mathcal{F}^W(t)/\mathcal{F}^W(t)$-measurable for every $t \in [0,T]$.

Now let $\lambda \in \mathcal{L}_{\mathrm{pl}}$ be given. We define $\bar{\lambda} \in \mathcal{L}_{\mathrm{pl}}$ by

$$\bar{\lambda}(\cdot,\varpi) \triangleq \lambda(\cdot,\varphi_\lambda^{-1}(\varpi)), \quad \forall\, \varpi \in \Omega, \tag{7.6}$$

and verify that (7.4) holds. According to Girsanov's theorem, if we impose on ω the measure $\mathbb{P}_\lambda$ given by (2.5), then $\varpi = \varphi_\lambda(\omega)$ is distributed according to Wiener measure $\mathbb{P}$. Therefore,

$$\begin{aligned}
&\mathbb{E}_\lambda\big[e^{-rT-\alpha\lambda(T)}g(S)\big]\\
&= \int_\Omega e^{-rT-\alpha\lambda(T,\omega)} g\big(S(0)\exp(\sigma W(\cdot,\omega)+\mu)\big)\,\mathbb{P}_\lambda(d\omega)\\
&= \int_\Omega e^{-rT-\alpha\bar{\lambda}(T,\varpi)} g\big(S(0)\exp(\sigma W(\cdot,\omega)+\lambda(\cdot,\omega)+\mu-\bar{\lambda}(\cdot,\varpi))\big)\,\mathbb{P}_\lambda(d\omega)\\
&= \int_\Omega e^{-rT-\alpha\bar{\lambda}(T,\varpi)} g\big(S(0)\exp(\sigma\varpi+\mu-\bar{\lambda}(\cdot,\varpi))\big)\,\mathbb{P}_\lambda(d\omega)\\
&= \int_\Omega e^{-rT-\alpha\bar{\lambda}(T,\varpi)} g\big(S(0)\exp(\sigma\varpi+\mu-\bar{\lambda}(\cdot,\varpi))\big)\,\mathbb{P}(d\varpi)\\
&= \int_\Omega e^{-rT-\alpha\bar{\lambda}(T,\varpi)} g\big(S(0)\exp(\sigma W(\cdot,\varpi)+\mu-\bar{\lambda}(\cdot,\varpi))\big)\,\mathbb{P}(d\varpi)\\
&= \mathbb{E}\big[e^{-rT-\alpha\bar{\lambda}(T)}g\big(Se^{-\bar{\lambda}}\big)\big],
\end{aligned}$$

which is (7.4).

For the converse, let $\bar{\lambda} \in \mathcal{L}_{\mathrm{pl}}$ be given. The function $\psi_{\bar{\lambda}} \colon \Omega \to \Omega$ defined by

$$\psi_{\bar{\lambda}}(\bar{\omega})(t) \triangleq \bar{\omega}(t) - \frac{1}{\sigma}\bar{\lambda}(t, \bar{\omega}), \quad t \in [0, T],\ \bar{\omega} \in \Omega$$

is bijective, and both it and its inverse are $\mathcal{F}^W(t)/\mathcal{F}^W(t)$-measurable for every $t \in [0, T]$ (it is formally merely $\varphi_{-\bar{\lambda}}$). Therefore, we can define $\lambda \in \mathcal{L}_{\mathrm{pl}}$ by

$$\lambda(\cdot, \omega) \triangleq \bar{\lambda}(\cdot, \psi_{\bar{\lambda}}^{-1}(\omega)), \quad \forall\, \omega \in \Omega. \tag{7.7}$$

According to the definitions and with the notation $\omega = \psi_{\bar{\lambda}}(\bar{\omega})$, we have

$$\varphi_{\lambda}(\omega) = \omega + \frac{1}{\sigma}\lambda(\cdot, \omega) = \psi_{\bar{\lambda}}(\bar{\omega}) + \frac{1}{\sigma}\bar{\lambda}(\cdot, \bar{\omega}) = \bar{\omega}.$$

In other words, φ_{λ} and $\psi_{\bar{\lambda}}$ are inverse functions and the relationship (7.7) between λ and $\bar{\lambda}$ coincides with the relationship (7.6). It follows that (7.4) again holds, and the theorem is proved. ♢

8 Proof of Theorem 5.3

We use the notation introduced in Section 5.

Lemma 8.1 *Given $g \colon C_+[0, T] \to [0, \infty)$, the map $g_* \colon C_+[0, T] \times R[0, T] \to [0, \infty)$ defined by* (5.3) *is lower semicontinuous in the second argument with respect to the topology of pointwise convergence on $R[0, T]$. If g is lower semicontinuous, then*

$$g_*(y, \lambda) = g(ye^{-\lambda}) \quad \forall\, (y, \lambda) \in C_+[0, T] \times (R[0, T] \cap C_+[0, T]). \tag{8.1}$$

Proof: To prove lower semicontinuity in the second argument, let $y \in C_+[0, T]$ and $\lambda \in R[0, T]$ be given, and let $\lambda_n \to \lambda$ pointwise, where each λ_n is in $R[0, T]$. Let $\varepsilon > 0$ be given. According to the definition of g_*, for each n we may choose $\eta_n \in R[0, T] \cap C_+[0, T]$ such that $g(ye^{-\eta_n}) \le \varepsilon + g_*(y, \lambda_n)$ and $d(\eta_n, \lambda_n) < \frac{1}{n}$. But then $\eta_n \to \lambda$ pointwise, which implies

$$g_*(y, \lambda) \le \liminf_{n \to \infty} g(ye^{-\eta_n}) \le \varepsilon + \liminf_{n \to \infty} g_*(y, \lambda_n).$$

Since $\varepsilon > 0$ is arbitrary, we have lower semicontinuity of $g_*(y, \cdot)$ at λ.

Assume now that g is lower semicontinuous and $(y, \lambda) \in C_+[0, T] \times (R[0, T] \cap C_+[0, T])$. Let $\lambda_n \to \lambda$ pointwise. Remark 5.1 shows that $\lambda_n \to \lambda$ uniformly. We have $g\big(ye^{-\lambda}\big) \le \liminf_{n \to \infty} g\big(ye^{-\lambda_n}\big)$, and minimizing over sequences $\{\lambda_n\}_{n=1}^{\infty}$ we obtain $g(ye^{-\lambda}) \le g_*(y, \lambda)$. Using the constant sequence $\{\lambda\}_{n=1}^{\infty}$, we obtain the reverse inequality. ♢

Proof of Theorem 5.3: Since $\mathcal{C} \subset \mathcal{R}$, the inequality

$$v(0, S(0); \alpha) \le \sup_{\lambda \in \mathcal{R}} \mathbb{E}\big[e^{-rT - \alpha\lambda(T)} g_*(S, \lambda)\big] \tag{8.2}$$

follows immediately from Theorem 3.1 and Lemma 8.1. For the reverse inequality, we show that

$$\sup_{\lambda\in\mathcal{C}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}g(Se^{-\lambda})\big] \geq \sup_{\lambda\in\mathcal{R}} \mathbb{E}\big[e^{-rT-\alpha\lambda(T)}g_*(S,\lambda)\big]. \tag{8.3}$$

To do this, we choose a process $\lambda \in \mathcal{R}$, and approximate it by a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of processes in $\mathcal{C}$.

Let $\lambda \in \mathcal{R}$ be given, and assume for the moment that $\lambda(T) \leq C$. We need to approximate λ by processes which are continuous. According to the definition of $\mathcal{R}$, there are finitely many pre-specified times $0 < t_1 < \cdots < t_I \leq T$ at which λ can be discontinuous. Denote the jumps of λ by

$$\alpha_i \triangleq \lambda(t_i) - \lambda(t_i-), \quad i = 1,\ldots,I,$$

and denote the continuous part of λ by

$$\lambda^{\mathrm{c}}(t) \triangleq \lambda(t) - \sum_{\{i;t_i\leq t\}} \alpha_i, \quad 0 \leq t \leq T.$$

Set $M_i(t) \triangleq \mathbb{E}[\alpha_i\,|\,\mathcal{F}(t)]$ for $0 \leq t \leq T$. Each M_i is a bounded, nonnegative martingale, relative to the Brownian filtration $\{\mathcal{F}(t); 0 \leq t \leq T\}$, and must therefore have a continuous modification ([12], Theorem 3.13 of Chapter 1 and Problem 4.16 of Chapter 3). Without loss of generality, we assume therefore that each M_i is continuous.

Choose $N \in \mathbb{N}$ so that $t_1 \geq 1/N$. For all $n \geq N$ and $t \in [0,T]$, define

$$\lambda_n(t) \triangleq \lambda^{\mathrm{c}}(t) + \sum_{i=1}^{I}\left[1 \wedge n\Big(t - t_i + \frac{1}{n}\Big)^{+}\right] \max_{t_i-\frac{1}{n}\leq s\leq t\wedge t_i} M_i(s).$$

Then λ_n is continuous, adapted, nondecreasing, and satisfies $\lambda_n(0) = 0$. If, in addition, n satisfies $t_{i+1} - t_i \geq 1/n$ for all $i \in \{1,\ldots,I-1\}$, then

$$\lambda_n(t_i) = \lambda^{\mathrm{c}}(t_i) + \sum_{j=1}^{i} \max_{t_j-\frac{1}{n}\leq s\leq t_j} M_j(s)$$

for all $i \in \{1,\ldots,I\}$; in particular,

$$\lim_{n\to\infty} \lambda_n(t_i) = \lambda^{\mathrm{c}}(t_i) + \sum_{j=1}^{i} \alpha_j = \lambda(t_i).$$

For $t \in [0,t_1)$ and sufficiently large n, we have $t \leq t_1 - 1/n$ and $\lambda_n(t) = \lambda^{\mathrm{c}}(t) = \lambda(t)$. For $t \in (t_i,t_{i+1})$, we have $t \leq t_{i+1} - 1/n$ for sufficiently large n, and then

$$\lambda_n(t) = \lambda^{\mathrm{c}}(t) + \sum_{j=1}^{i} \max_{t_j-\frac{1}{n}\leq s\leq t_j} M_j(s) \to \lambda^{\mathrm{c}}(t) + \sum_{j=1}^{i} \alpha_j = \lambda(t).$$

In other words, $\lambda_n \in \mathcal{C}$ and $\lambda_n \to \lambda$ pointwise almost surely.

We relax the condition that $\lambda(T) \le C$. Assuming only that $\lambda(T) < \infty$ almost surely, we define for each $m \in \mathbb{N}$ the process $\lambda_m = m \wedge \lambda$. We have just proved that for each m we may construct a sequence $\{\lambda_{m,n}\}_{n=N}^{\infty}$ in $\mathcal{C}$ such that $\lambda_{m,n} \to \lambda_m$ pointwise almost surely as $n \to \infty$. Let d be the metric of pointwise convergence on $R[0,T]$ defined in Section 5. We may choose a subsequence $\{\lambda_{m_k}\}_{k=1}^{\infty}$ such that $\mathbb{P}\big\{d(\lambda_{m_k}, \lambda) \ge 1/k\big\} \le 2^{-k}$ for all $k \in \mathbb{N}$, and then choose n_k such that $\mathbb{P}\big\{d(\lambda_{m_k,n_k}, \lambda_{m_k}) \ge 1/k\big\} \le 2^{-k}$. Then

$$\mathbb{P}\left\{d(\lambda_{m_k,n_k}, \lambda) \ge \frac{2}{k}\right\} \le \mathbb{P}\left\{d(\lambda_{m_k,n_k}, \lambda_{m_k}) \ge \frac{1}{k} \text{ or } d(\lambda_{m_k}, \lambda) \ge \frac{1}{k}\right\} \le 2^{-k+1}.$$

The Borel-Cantelli Lemma implies $\mathbb{P}\big\{d(\lambda_{m_k,n_k}, \lambda) \ge 2/k \text{ infinitely often}\big\} = 0$, and hence $\lambda_{m_k,n_k} \to \lambda$ pointwise almost surely as $k \to \infty$.

In either case, whether $\lambda \in \mathcal{R}$ is bounded or not, there is a sequence $\{\lambda_n\}_{n=N}^{\infty}$ of processes in $\mathcal{C}$ such that $\lambda_n \to \lambda$ pointwise almost surely. For this sequence, we have from Fatou's Lemma and the lower semicontinuity of $g_*(S,\cdot)$ that

$$\begin{aligned}\sup_{\eta \in \mathcal{C}} \mathbb{E}\big[e^{-rT-\alpha\eta(T)} g(Se^{-\eta})\big] &\ge \liminf_{n\to\infty} \mathbb{E}\big[e^{-rT-\alpha\lambda_n(T)} g\big(Se^{-\lambda_n}\big)\big] \\ &\ge \mathbb{E}\Big[\liminf_{n\to\infty} e^{-rT-\alpha\lambda_n(T)} g\big(Se^{-\lambda_n}\big)\Big] \\ &\ge \mathbb{E}\big[e^{-rT-\alpha\lambda(T)} g_*(S,\lambda)\big].\end{aligned}$$

Taking the supremum of the right-hand side over $\lambda \in \mathcal{R}$, we obtain (8.3). $\diamondsuit$

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