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Centre for Practical Quantitative Finance

CPQF Working Paper Series
No. 20

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April 2010

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Version 1: September, 8th 2009

Version 2: March, 20th 2010

Abstract The foreign exchange options market is one of the largest and most liquid OTC derivative markets in the world. Surprisingly, very little is known in the academic literature about the construction of the most important object in this market: The implied volatility smile. The smile construction procedure and the volatility quoting mechanisms are FX specific and differ significantly from other markets. We give a detailed overview of these quoting mechanisms and introduce the resulting smile construction problem. Furthermore, we provide a new formula which can be used for an efficient and robust FX smile construction.

Keywords: FX Quotations, FX Smile Construction, Risk Reversal, Butterfly, Strangle, Delta Conventions, Malz Formula

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1 Delta– and ATM–Conventions in FX-Markets

1.1 Introduction

It is common market practice to summarize the information of the vanilla options market in the volatility smile table which includes Black-Scholes implied volatilities for different maturities and moneyness levels. The degree of moneyness of an option can be represented by the strike or any linear or non-linear transformation of the strike (forward-moneyness, log-moneyness, delta). The implied volatility as a function of moneyness for a fixed time to maturity is generally referred to as the smile. The volatility smile is the crucial object in pricing and risk management procedures since it is used to price vanilla, as well as exotic option books. Market participants entering the FX OTC derivative market are confronted with the fact that the volatility smile is usually not directly observable in the market. This is in opposite to the equity market, where strike-price or strike-volatility pairs can be observed. In foreign exchange OTC derivative markets it is common to publish currency pair specific risk reversal σ_{RR}, strangle σ_{STR} and at-the-money volatility σ_{ATM} quotes as given in the market sample in Table 1. These quotes can be used to construct a

Table 1: FX Market data for a maturity of 1 month, as of January, 20th 2009

	EURUSD	USDJPY
σ_{ATM}	21.6215%	21.00%
σ_{RR}	−0.5%	−5.3%
σ_{STR}	0.7375%	0.184%

complete volatility smile from which one can extract the volatility for any strike. In the next section we will introduce the basic FX terminology which is necessary to understand the following sections. We will then explain the market implied information for quotes such as those given in Table 1. Finally, we will propose an implied volatility function which accounts for this information.

1.2 Spot, Forward and Vanilla Options

FX Spot Rate S_t

The FX spot rate S_t =FOR-DOM represents the number of units of domestic currency needed to buy one unit of foreign currency at time t. For example, EUR-USD= 1.3900 means that one EUR is worth 1.3900 USD. In this case, EUR is the foreign currency and USD is the domestic one. The EUR-USD= 1.3900 quote is equivalent to USD-EUR 0.7194. A notional of N units of foreign currency is equal to NS_t units of domestic currency (see also Wystup (2006)). The term "domestic" does *not* refer to any geographical region. The domestic currency is also referred to as the numeraire currency (see Castagna (2010)).

FX Outright Forward Rate $f(t,T)$

By far the most popular and liquid hedge contract for a corporate treasurer is the outright forward contract. This contract trades at time t at zero cost and leads to an exchange of notionals at time T at the pre-specified outright forward rate $f(t,T)$. At time T, the foreign notional amount N would be exchanged against an amount of $Nf(t,T)$ domestic currency units. The outright forward is related to the FX spot rate via the spot-rates parity

$$f(t,T) = S_t \cdot e^{(r_d - r_f)\tau}\ , \tag{1}$$

where

r_f is the foreign interest rate (continuously compounded),
r_d is the domestic interest rate (continuously compounded),
τ is the time to maturity, equal to $T-t$.

FX Forward Value

At inception an outright forward contract has a value of zero. Thereafter, when markets move, the value of the forward contract is no longer zero but is worth

$$v_f(t,T) = e^{-r_d\tau}\ (f(t,T) - K) = S_t e^{-r_f\tau} - K e^{-r_d\tau} \tag{2}$$

for a pre-specified exchange rate K. This is the forward contract value in domestic currency units, marked to the market at time t.

FX Vanilla Options

In foreign exchange markets options are usually physically settled, i.e. the buyer of a EUR vanilla call (USD Put) receives a EUR notional amount N and pays $N \times K$ USD, where K is the strike. The value of such a vanilla contract is computed with the standard Black-Scholes formula

$$\begin{aligned} v(S_t,K,\sigma,\phi) &= v(S_t,r_d,r_f,K,\sigma,t,T,\phi) \\ &= \phi[e^{-r_f\tau}\, S_t\, N(\phi d_+)\ -\ e^{-r_d\tau}\, K\, N(\phi d_-)] \end{aligned} \tag{3}$$

$$= \phi e^{-r_d \tau} \left[f(t,T)\, N(\phi d_+) - K\, N(\phi d_-) \right],$$

where

$$d_\pm = \frac{\ln\left(\frac{f(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

K : strike of the FX option,

σ : Black-Scholes volatility,

$\phi = +1$ for a call , $\phi = -1$ for a put,

$N(x)$: cumulative normal distribution function.

We may drop some of the variables of the function v depending on context. The Black-Scholes formula renders a value v in domestic currency. An equivalent value of the position in foreign currency is v/S_t. The accounting currency (the currency in which the option values are measured) is also called the **premium currency**. The **notional** is the amount of currency which the holder of an option is entitled to exchange. The value formula applies by default to one unit of foreign notional (corresponding to one share of stock in equity markets), with a value in units of domestic currency. An example which illustrates these terms follows. Consider a EUR-USD call with a spot of $S_0 = 1.3900$, a strike of $K = 1.3500$ and a price of 0.1024 USD. If a notional of $1,000,000$ EUR is specified, the holder of the option will receive $1,000,000$ EUR and pay $1,350,000$ USD at maturity and the option's current price is $102,400$ USD ($73,669$ EUR).

1.3 Delta Types

The delta of an option is the percentage of the foreign notional one must buy when selling the option to hold a hedged position (equivalent to buying stock). For instance, a delta of $0.35 = 35\%$ indicates buying 35% of the foreign notional to delta-hedge a short option. In foreign exchange markets we distinguish the cases *spot delta* for a hedge in the spot market and *forward delta* for a hedge in the FX forward market. Furthermore, the standard delta is a quantity in percent of foreign currency. The actual hedge quantity must be changed if the premium is paid in foreign currency, which would be equivalent to paying for stock options in shares of stock. We call this type of delta the *premium-adjusted delta*. In the previous example the value of an option with a notional of $1,000,000$ EUR was calculated as $73,669$ EUR. Assuming a short position with a delta of 60% means, that buying $600,000$ EUR is necessary to hedge. However the final hedge quantity will be $526,331$ EUR which is the delta quantity reduced by the received premium in EUR. Consequently, the premium-adjusted delta would be 52.63%. The following sections will introduce the formulas for the different delta types. A detailed introduction on at-the-money and delta conventions which we used as an orientation can be found in Beier and Renner (2010). Related work, which is worth reading and describes the standard conventions can also be found in Beneder and Elkenbracht-Huizing (2003), Bossens et al. (2009), Castagna (2010), Clark (forthcoming).

Unadjusted Deltas

Spot Delta

The sensitivity of the vanilla option with respect to the spot rate S_t is given as

$$\Delta_S(K,\sigma,\phi) \stackrel{\Delta}{=} \frac{\partial v}{\partial S} = v_S. \tag{4}$$

Standard calculus yields

$$\Delta_S(K,\sigma,\phi) = \phi e^{-r_f \tau} N(\phi d_+), \tag{5}$$

$$\text{Put-call delta parity:} \quad \Delta_S(K,\sigma,+1) - \Delta_S(K,\sigma,-1) = e^{-r_f \tau}. \tag{6}$$

In equity markets, one would buy Δ_S units of the stock to hedge a short vanilla option position. In FX markets, this is equivalent to buying Δ_S times the foreign notional N. This is equivalent to selling of $\Delta_S \times N \times S_t$ units of domestic currency. Note that the absolute value of delta is a number between zero and a discount factor $e^{-r_f \tau} < 100\%$. Therefore, 50% is not the center value for the delta range.

Forward Delta

An alternative to the spot hedge is a hedge with a forward contract. The number of forward contracts one would buy in this case differs from the number of units in a spot hedge. The forward-hedge ratio is given by

$$\Delta_f(K,\sigma,\phi) \stackrel{\Delta}{=} \frac{\partial v}{\partial v_f} = \frac{\partial v}{\partial S}\frac{\partial S}{\partial v_f} = \frac{\partial v}{\partial S}\left(\frac{\partial v_f}{\partial S}\right)^{-1} = \phi N(\phi d_+), \tag{7}$$

$$\text{Put-call delta parity:} \quad \Delta_f(K,\sigma,+1) - \Delta_f(K,\sigma,-1) = 100\%. \tag{8}$$

In the hedge, one would enter $\Delta_f \times N$ forward contracts to forward-hedge a short vanilla option position. The forward delta is often used in FX options smile tables, because of the fact that the delta of a call and the (absolute value of the) delta of the corresponding put add up to 100%, i.e. a 25-delta call must have the same volatility as a 75-delta put. This symmetry only works for forward deltas.

Premium Adjusted Deltas

Premium-Adjusted Spot Delta

The premium-adjusted spot delta takes care of the correction induced by payment of the premium in foreign currency, which is the amount by which the delta hedge in foreign currency has to be corrected. The delta can be represented as

$$\Delta_{S,pa} \stackrel{\Delta}{=} \Delta_S - \frac{v}{S}. \tag{9}$$

In this hedge scenario, one would buy $N(\Delta_S - \frac{v}{S_t})$ foreign currency units to hedge a short vanilla position. The equivalent number of domestic currency units to sell is $N(S_t\Delta_S - v)$. To quantify the hedge in domestic currency we need to flip around the quotation and compute the *dual delta*

$$\begin{aligned}
\frac{\partial \frac{v}{S} \text{ in FOR}}{\partial \frac{1}{S} \text{ in FOR per DOM}} &= \text{ DOM to buy} \\
&= \frac{\partial \frac{v}{S}}{\partial S} \cdot \frac{\partial S}{\partial \frac{1}{S}} \\
&= \frac{Sv_S - v}{S^2} \cdot \left(\frac{\partial \frac{1}{S}}{\partial S} \right)^{-1} = \frac{Sv_S - v}{S^2} \cdot \left(-\frac{1}{S^2} \right)^{-1} \\
&= -(Sv_S - v) \text{ DOM to buy} = Sv_S - v \quad \text{DOM to sell} = v_S - \frac{v}{S} \quad \text{FOR to buy},
\end{aligned}$$

which confirms the definition of the premium-adjusted delta in Equation (9). We find

$$\Delta_{S,pa}(K,\sigma,\phi) = \phi e^{-r_f\tau} \frac{K}{f} N(\phi d_-), \tag{10}$$

$$\text{Put-call delta parity: } \Delta_{S,pa}(K,\sigma,+1) - \Delta_{S,pa}(K,\sigma,-1) = e^{-r_f\tau} \frac{K}{f}. \tag{11}$$

Note that

$$\begin{aligned}
\phi e^{-r_f\tau} \frac{K}{f} N(\phi d_-) &= \text{ FOR to buy per 1 FOR} \\
\phi e^{-r_d\tau} K N(\phi d_-) &= \text{ DOM to sell per 1 FOR} \\
-\phi e^{-r_d\tau} K N(\phi d_-) &= \text{ DOM to buy per 1 FOR} \\
-\phi e^{-r_d\tau} N(\phi d_-) &= \text{ DOM to buy per 1 DOM} \\
= v_K &= \text{ the dual delta},
\end{aligned}$$

which is the strike-coefficient in the Black-Scholes Formula (3). It is now apparent that this can also be interpreted as a delta, the spot delta in reverse quotation DOM-FOR. For the premium-adjusted delta the relationship strike versus delta is not injective: for a given delta there might exist more than one corresponding strike. This is shown in Figure (1).

Premium-Adjusted Forward Delta

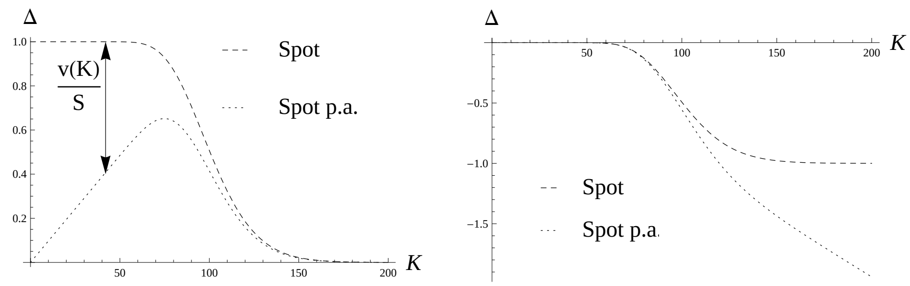

Fig. 1: Premium-adjusted and standard call (left chart) and put (right chart) spot delta, $S_t = 100$, $\tau = 1.0$, $r_d = 0.03$, $r_f = 0.0$, $\sigma = 0.2$.

As in the case of a spot delta, a premium payment in foreign currency leads to an adjustment of the forward delta. The resulting hedge quantity is given by

$$\Delta_{f,pa}(K,\sigma,\phi) = \phi \frac{K}{f} N(\phi d_-), \tag{12}$$

$$\text{Put-call delta parity:} \quad \Delta_{f,pa}(K,\sigma,+1) - \Delta_{f,pa}(K,\sigma,-1) = \frac{K}{f}. \tag{13}$$

Note again that the premium-adjusted forward delta of a call is not a monotone function of the strike.

Delta Conventions for Selected Currency Pairs

This section is based on Ian Clark's summary of current FX market conventions (see Clark (forthcoming)). The question which of the deltas is used in practice cannot be answered systematically. Both, spot and forward deltas are used, depending on which product is used to hedge. Generally, forward hedges are popular to capture rates risk besides the spot risk. So naturally, forward hedges come up for delta-one-similar products or for long-term options. In practice, the immediate hedge executed is generally the spot-hedge, because it has to be done instantaneously with the option trade. At a later time the trader can change the spot hedge to a forward hedge using a zero-cost FX swap.

Forward delta conventions are normally used to specify implied volatilities because of the symmetry of put and call deltas adding up to 100%. Using forward deltas as a quotation standard often depends on the time to expiry T and the presence of an emerging market currency in the currency pair. If the currency pair does contain an emerging market currency, forward deltas are the market default. If the currency pair contains only currencies from the OECD economies (USD, EUR, JPY,

GBP, AUD, NZD, CAD, CHF, NOK, SEK, DKK), and does not include ISK, TRY, MXN, CZK, KRW, HUF, PLN, ZAR or SGD, then spot deltas are used out to and including 1Y. For all longer dated tenors forward deltas are used. An example: the NZD-JPY uses spot deltas for maturities below 1 year and forward deltas for maturities above 1 year. However, for the CZK-JPY currency pair forward deltas are used in the volatility smile quotation(see Clark (forthcoming)). The premium-adjusted delta as a default is used for options in currency pairs whose premium currency is FOR. We provide examples in Table 2. The market standard is to take the more

Table 2: Selected currency pairs and their default premium currency determining the delta type. Source: Clark (forthcoming)

Currency Pair	Premium Currency	Delta Convention
EUR-USD	USD	regular
USD-JPY	USD	premium-adjusted
EUR-JPY	EUR	premium-adjusted
USD-CHF	USD	premium-adjusted
EUR-CHF	EUR	premium-adjusted
GBP-USD	USD	regular
EUR-GBP	EUR	premium-adjusted
AUD-USD	USD	regular
AUD-JPY	AUD	premium-adjusted
USD-CAD	USD	premium-adjusted
USD-BRL	USD	premium-adjusted
USD-MXN	USD	premium-adjusted

commonly traded currency as the premium currency. However, this does not apply to the JPY. Virtually all currency pairs involving the USD will have the USD as the premium currency of the contract. Similarly, contracts on a currency pair including the EUR - and not the USD - will be denoted in EUR. A basic premium currency hierarchy is given as (Clark (forthcoming))

$$\begin{aligned} \text{USD} &\succ \text{EUR} \succ \text{GBP} \succ \text{AUD} \succ \text{NZD} \succ \text{CAD} \succ \text{CHF} \\ &\succ \text{NOK,SEK,DKK} \\ &\succ \text{CZK,PLN,TRY,MXN} \succ \text{JPY} \succ \dots \end{aligned} \tag{14}$$

Exceptions may occur, so in case of doubt it is advisable to check.

1.4 At-The-Money Definitions

Defining *at-the-money (ATM)* is by far not as obvious as one might think when first studying options. It is the attempt to specify the middle of the spot distribution in various senses. We can think of

ATM-spot	$K = S_0$
ATM-fwd	$K = f$
ATM-value-neutral	K such that call value = put value
ATM-Δ-neutral	K such that call delta = − put delta.

In addition to that, the notion of ATM involving delta will have sub-categories depending on which delta convention is used. **ATM-spot** is often used in beginners' text books or on term sheets for retail investors, because the majority of market participants is familiar with it. **ATM-fwd** takes into account that the risk-neutral expectation of the future spot is the forward price (1), which is a natural way of specifying the "middle". It is very common for currency pairs with a large interest rate differential (emerging markets) or long maturity. **ATM-value-neutral** is equivalent to **ATM-fwd** because of the put-call parity. Choosing the strike in the **ATM-delta-neutral** sense ensures that a straddle with this strike has a zero spot exposure which accounts for the traders' vega-hedging needs. This ATM convention is considered as the default ATM notion for short-dated FX options. We summarize the various at-the-money definitions and the relations between all relevant quantities in Table 3.

Table 3: ATM Strike values and delta values for the different delta conventions. Source: Beier and Renner (2010)

	ATM Δ-neutral Strike	ATM fwd Strike	ATM Δ-neutral Delta	ATM fwd Delta
Spot Delta	$f\,e^{\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi e^{-r_f\tau}$	$\phi e^{-r_f\tau}\,N(\phi\frac{1}{2}\sigma\sqrt{\tau})$
Forward Delta	$f\,e^{\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi$	$\phi N(\phi\frac{1}{2}\sigma\sqrt{\tau})$
Spot Delta p.a.	$f\,e^{-\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi e^{-r_f\tau}\,e^{-\frac{1}{2}\sigma^2\tau}$	$\phi e^{-r_f\tau}\,N(-\phi\frac{1}{2}\sigma\sqrt{\tau})$
Forward Delta p.a.	$f\,e^{-\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi e^{-\frac{1}{2}\sigma^2\tau}$	$\phi N(-\phi\frac{1}{2}\sigma\sqrt{\tau})$

1.5 Delta–Strike Conversion

Professional FX market participants have adapted specific quoting mechanisms which differ significantly from other markets. While it is common in equity markets to quote strike-volatility or strike-price pairs, this is usually not the case in FX markets. Many customers on the buy-side receive implied volatility-delta pairs from their market data provider. This data is usually the result of a suitable calibration and transformation output. The calibration is based on data which has the type shown in Table 1. The market participant is then confronted with the task to transform volatility-delta to strike-price pairs respecting FX specific at-the-money and delta definitions. This section will outline the algorithms which can be used to that end. For a given spot delta Δ_S and the corresponding volatility σ the strike can be retrieved with

$$K = f e^{-\phi N^{-1}(\phi e^{r_f \tau} \Delta_S)\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau}. \tag{15}$$

The equivalent forward delta version is

$$K = f e^{-\phi N^{-1}(\phi \Delta_f)\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau}. \tag{16}$$

Conversion of a Premium-Adjusted Forward Delta to Strike

For a premium-adjusted forward delta the relationship between delta and strike

$$\Delta_{f,pa}(K,\sigma,\phi) \;=\; \phi\frac{K}{f}N(\phi d_-) \;=\; \phi\frac{K}{f}N\left(\phi\frac{\ln\left(\frac{f}{K}\right)-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right),$$

can not be solved for the strike in closed form. A numerical procedure has to be used. This is straightforward for the put delta because the put delta is monotone in strike. This is not the case for the premium-adjusted call delta, as illustrated in Figure (1). Here, two strikes can be obtained for a given premium-adjusted call delta (for example for $\Delta_{S,pa} = 0.2$). It is common to search for strikes corresponding to deltas which are on the right hand side of the delta maximum. This is illustrated as a shadowed area in the left chart of Figure (2).

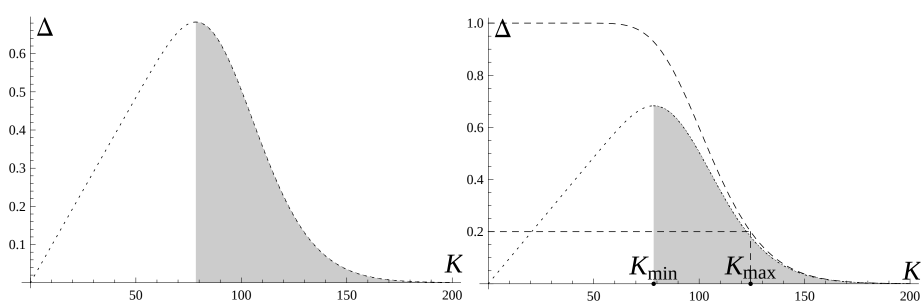

Fig. 2: Strike region for given premium-adjusted delta. $S_t = 100$

Consequently, we recommend to use Brent's root searcher (see Brent (2002)) to search for $K \in [K_{min}, K_{max}]$. The right limit K_{max} can be chosen as the strike corresponding to the non premium-adjusted delta, since the premium-adjusted delta for a strike K is always smaller than the simple delta corresponding to the same strike. For example, if we are looking for a strike corresponding to a premium-adjusted forward delta of 0.20, we can choose K_{max} to be the strike corresponding to a simple forward delta of 0.20. The last strike can be calculated analytically using Equation (16). It is easy to see that the premium-adjusted delta is always below the non-premium-adjusted one. This follows from

$$\begin{aligned}\Delta_S(K,\sigma,\phi) - \Delta_{S,pa}(K,\sigma,\phi) = e^{-r_f\tau}\phi N(\phi d_+) - \phi e^{-r_f\tau}\frac{K}{f}N(\phi d_-) \geq 0 \\ \Leftrightarrow \phi f N(\phi d_+) - \phi K N(\phi d_-) \geq 0.\end{aligned}$$

Discounting the last inequality yields the Black-Scholes formula, which is always positive. The maximum for both, the premium-adjusted spot and premium-adjusted forward delta, is given implicitly by the equation

$$\sigma\sqrt{\tau}N(d_-) = n(d_-),$$

with $n(x)$ being the normal density at x. One can solve this implicit equation numerically for K_{min} and then use Brent's method to search for the strike in $[K_{min}, K_{max}]$. The resulting interval is illustrated in the right hand side of Figure (2).

Construction of Implied Volatility Smiles

The previous section introduced the FX specific delta and ATM conventions. This knowledge is crucial to understand the volatility construction procedure in FX markets. In FX option markets it is common to use the delta to measure the degree of moneyness. Consequently, volatilities are assigned to deltas (for any delta type), rather than strikes. For example, it is common to quote the volatility for an option which has a premium-adjusted delta of 0.25. These quotes are often provided by market data vendors to their customers. However, the volatility-delta version of the smile is translated by the vendors after using the smile construction procedure discussed below. Other vendors do not provide delta-volatility quotes. In this case, the customers have to employ the smile construction procedure. Related sources covering this subject can be found in Bossens et al. (2009), Castagna (2010), Clark (forthcoming).

Unlike in other markets, the FX smile is given implicitly as a set of restrictions implied by market instruments. This is FX-specific, as other markets quote volatility versus strike directly. A consequence is that one has to employ a calibration procedure to construct a volatility vs. delta or volatility vs. strike smile. This section introduces the set of restrictions implied by market instruments and proposes a new method which allows an efficient and robust calibration.

Suppose the mapping of a strike to the corresponding implied volatility

$$K \mapsto \sigma(K)$$

is given. We will not specify $\sigma(K)$ here but treat it as a general smile function for the moment. The crucial point in the construction of the FX volatility smile is to build $\sigma(K)$ such that it matches the volatilities and prices implied by market quotes. The FX market uses three volatility quotes for a given delta such as $\Delta = \pm 0.25$ [1]:

- an at-the-money volatility σ_{ATM},
- a risk reversal volatility σ_{25-RR},
- a quoted strangle volatility σ_{25-S-Q}.

A sample of market quotes for the EURUSD and USDJPY currency pairs is given in Table 4. Before starting the smile construction it is important to analyze the exact

Table 4: Market data for a maturity of 1 month, as of January, 20th 2009

	EURUSD	USDJPY
S_0	1.3088	90.68
r_d	0.3525%	0.42875%
r_f	2.0113%	0.3525%
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	−0.5%	−5.3%
σ_{25-S-Q}	0.7375%	0.184%

characteristics of the quotes in Table 4. In particular, one has to identify first

- which at-the-money convention is used,
- which delta type is used.

For example, Figure (3) shows two market consistent smiles based on the EURUSD market data from Table 4, assuming that this data refers to different deltas, a simple **or** premium-adjusted one. It is obvious, that the smiles can have very different shapes, in particular for out-of-the-money and in-the-money options. Misunderstanding the delta type which the market data refers to would lead to a wrong pricing of vanilla options. The quotes in the given market sample refer to a spot delta for the currency pair EURUSD and a premium-adjusted spot delta for the currency pair USDJPY. Both currency pairs use the forward delta neutral at-the-money quotation. The next subsections explain which information these quotes contain.

At-the-Money Volatility

After identifying the at-the-money type, we can extract the at-the-money strike K_{ATM} as summarized in Table 3. For the market sample data in Table 4 the corresponding strikes are summarized in Table 5. Independent of the choice of $\sigma(K)$,

[1] We will take a delta of 0.25 as an example, although any choice is possible.

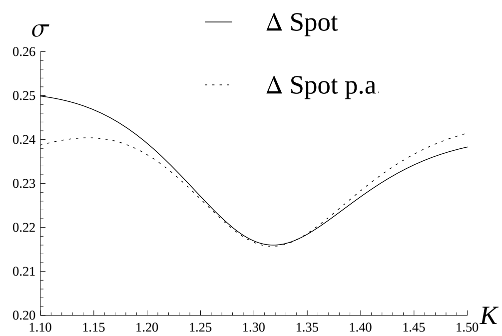

Fig. 3: Smile construction with EURUSD market data from Table 4, assuming different delta types.

Table 5: At-the-money strikes for market sample

	EURUSD	USDJPY
K_{ATM}	1.3096	90.86

it has to be ensured that the volatility for the at-the-money strike is σ_{ATM}. Consequently, the construction procedure for $\sigma(K)$ has to guarantee that the following Equation

$$\sigma(K_{ATM}) = \sigma_{ATM} \tag{17}$$

holds. A market consistent smile function $\overline{\sigma}(K)$ for the EURUSD currency pair thus has to yield

$$\overline{\sigma}(1.3096) = 21.6215\%$$

for the market data in Table 4. We will show later how to calibrate $\sigma(K)$ to retrieve $\overline{\sigma}(K)$, so assume for the moment that the calibrated, market consistent smile function $\overline{\sigma}(K)$ is given.

Risk Reversal

The risk reversal quotation σ_{25-RR} is the difference between two volatilities:

- the implied volatility of a call with a delta of 0.25 and
- the implied volatility of a put with a delta of -0.25.

It measures the skewness of the smile, the extra volatility which is added to the 0.25Δ put volatility compared to a call volatility which has the same absolute delta.

Clearly, the delta type has to be specified in advance. For example, the implied volatility of a USD call JPY put with a premium-adjusted spot delta of 0.25 could be considered. Given $\sigma(K)$, it is possible to extract strike-volatility pairs[2] for a call and a put

$$\Big(K_{25C},\sigma(K_{25C})\Big) \qquad \Big(K_{25P},\sigma(K_{25P})\Big)$$

which yield a delta of 0.25 and -0.25 respectively:

$$\begin{aligned}\Delta\ (K_{25C},\sigma(K_{25C}),1) &= 0.25\\ \Delta\ (K_{25P},\sigma(K_{25P}),-1) &= -0.25\end{aligned}$$

In the equation system above, Δ denotes a general delta which has to be specified to $\Delta_S,\Delta_{S,pa}$ or $\Delta_f,\Delta_{f,pa}$. The market consistent smile function $\overline{\sigma}(K)$ has to match the information implied in the risk reversal. Consequently, it has to fulfill

$$\overline{\sigma}(K_{25C})-\overline{\sigma}(K_{25P})=\sigma_{25-RR}. \tag{18}$$

Examples of such 0.25 Δ strike-volatility pairs for the market data in Table 4 and a calibrated smile function $\overline{\sigma}(K)$ are given in Table 6.

For the currency pair EURUSD we can calculate the difference of the 0.25 Δ call

Table 6: 0.25 Δ strikes

	EURUSD	USDJPY
K_{25C}	1.3677	94.10
K_{25P}	1.2530	86.51
$\overline{\sigma}(K_{25C})$	22.1092%	18.7693%
$\overline{\sigma}(K_{25P})$	22.6092%	24.0693%

and put volatilities as

$$\overline{\sigma}(1.3677)-\overline{\sigma}(1.2530)=22.1092\%-22.6092\%=-0.5\%$$

which is consistent with the risk reversal quotation in Table 4. It can also be verified that

$$\Delta_S\ (1.3677,22.1092\%,1)=0.25 \text{ and } \Delta_S(1.2530,22.6092\%,-1)=-0.25.$$

[2] This can be achieved by using a standard root search algorithm.

Market Strangle

The strangle is the third restriction on the function $\sigma(K)$. Define the market strangle volatility σ_{25-S-M} as

$$\sigma_{25-S-M} = \sigma_{ATM} + \sigma_{25-S-Q}. \tag{19}$$

For the market sample from Table 4 and the USDJPY case this would correspond to

$$\sigma_{25-S-M} = 21.00\% + 0.184\% = 21.184\%.$$

Given this single volatility, we can extract a call strike $K_{25C-S-M}$ and a put strike $K_{25P-S-M}$ which - using σ_{25-S-M} as the volatility - yield a delta of 0.25 and -0.25 respectively. The procedure to extract a strike given a delta and volatility has been introduced in Section 1.5. The resulting strikes will then fulfill

$$\Delta\ (K_{25C-S-M}, \sigma_{25-S-M}, 1) = 0.25 \tag{20}$$

$$\Delta\ (K_{25P-S-M}, \sigma_{25-S-M}, -1) = -0.25. \tag{21}$$

The strikes corresponding to the market sample are summarized in Table 7. For the USDJPY case the strike volatility combinations given in Table 7 fulfill

$$\Delta_{S,pa}(94.55, 21.184\%, 1) = 0.25 \tag{22}$$

$$\Delta_{S,pa}(87.00, 21.184\%, -1) = -0.25 \tag{23}$$

where $\Delta_{S,pa}(K, \sigma, \phi)$ is the premium-adjusted spot delta. Given the strikes $K_{25C-S-M}$, $K_{25P-S-M}$ and the volatility σ_{25-S-M}, one can calculate the price of an option position of a long call with a strike of $K_{25C-S-M}$ and a volatility of σ_{25-S-M} and a long put with a strike of $K_{25P-S-M}$ and the same volatility. The resulting price v_{25-S-M} is

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1) \tag{24}$$

and is the final variable one is interested in. This is the third information implied by the market: The sum of the call option with a strike of $K_{25C-S-M}$ and the put option with a strike of $K_{25P-S-M}$ has to be v_{25-S-M}. This information has to be incorporated by a market consistent volatility function $\overline{\sigma}(K)$ which can have different volatilities at the strikes $K_{25C-S-M}$, $K_{25P-S-M}$ but should guarantee that the corresponding option prices at these strikes add up to v_{25-S-M}. The delta of these options with the smile volatilities is not restricted to yield 0.25 or -0.25. To summarize,

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma(K_{25C-S-M}), 1) + v(K_{25P-S-M}, \sigma(K_{25P-S-M}), -1) \tag{25}$$

is the last restriction on the volatility smile. Taking again the USDJPY as an example yields that the strangle price to be matched is

$$v_{25-S-M} = v(94.55, 21.184\%, 1) + v(87.00, 21.184\%, -1) = 1.67072. \tag{26}$$

The resulting price v_{25-S-M} is in the domestic currency, JPY in this case. One can then extract the volatilities from a calibrated smile $\overline{\sigma}(K)$ –as in Table 7– and calculate the strangle price with volatilities given by the calibrated smile function $\overline{\sigma}(K)$

$$v(94.55, 18.5435\%, 1) + v(87.00, 23.7778\%, -1) = 1.67072. \tag{27}$$

This is the same price as the one implied by the market in Equation (26).

Table 7: Market Strangle data

	EURUSD	USDJPY
$K_{25C-S-M}$	1.3685	94.55
$K_{25P-S-M}$	1.2535	87.00
$\overline{\sigma}(K_{25C-S-M})$	22.1216%	18.5435%
$\overline{\sigma}(K_{25P-S-M})$	22.5953%	23.7778%
v_{25-S-M}	0.0254782	1.67072

The introduced smile construction procedure is designed for a market that quotes three volatilities. This is often the case for illiquid markets. It can also be used for markets where more than three volatilities are quoted on an irregular basis, such that these illiquid quotes might not be a necessary input.

The Simplified Formula

Very often, a simplified formula is stated in the literature which allows an easy calculation of the 0.25 delta volatilities given the market quotes. Let σ_{25C} be the call volatility corresponding to a delta of 0.25 and σ_{25P} the -0.25 delta put volatility. Let K_{25C} and K_{25P} denote the corresponding strikes. The simplified formula states that

$$\begin{aligned} \sigma_{25C} &= \sigma_{ATM} + \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-Q} \\ \sigma_{25P} &= \sigma_{ATM} - \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-Q}. \end{aligned} \tag{28}$$

This would allow a simple calculation of the 0.25Δ volatilities $\sigma_{25C}, \sigma_{25P}$ with market quotes as given in Table 4. Including the at-the-money volatility would result in a smile with three anchor points which can then be interpolated in the usual way. In this case, no calibration procedure is needed. Note, that

$$\sigma_{25C} - \sigma_{25P} = \sigma_{25-RR} \tag{29}$$

such that the 0.25 Δ volatility difference automatically matches the quoted risk reversal volatility. The simplified formula can be reformulated to calculate σ_{25-S-Q}, given σ_{25C}, σ_{25P} and σ_{ATM} quotes. This yields

$$\sigma_{25-S-Q} = \frac{\sigma_{25C} + \sigma_{25P}}{2} - \sigma_{ATM}, \tag{30}$$

which presents the strangle as a convexity parameter. However, the problem arises in the matching of the market strangle as given in Equation (24), which we repeat here for convenience

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1).$$

Interpolating the smile from the three anchor points given by the simplified formula and calculating the market strangle with the corresponding volatilities at $K_{25P-S-M}$ and $K_{25C-S-M}$ does not necessary lead to the matching of v_{25-S-M}. The reason why the formula is stated very often (see for example Malz (1997)) is that the market strangle matching works for small risk reversal volatilities σ_{25-RR}. Assume that σ_{25-RR} is zero. The simplified Formula (28) then reduces to

$$\begin{aligned} \sigma_{25C} &= \sigma_{ATM} + \sigma_{25-S-Q}, \\ \sigma_{25P} &= \sigma_{ATM} + \sigma_{25-S-Q}. \end{aligned}$$

This implies, that the volatility corresponding to a delta of 0.25 is the same as the volatility corresponding to a delta of -0.25, which is the same as the market strangle volatility σ_{25-S-M} introduced in Equation (19). Assume that in case of a vanishing risk reversal the smile is built using three anchor points given by the simplified formula and one is asked to price a strangle with strikes $K_{25C-S-M}$ and $K_{25P-S-M}$. Given the volatility $\sigma_{25C} = \sigma_{ATM} + \sigma_{25-S-Q}$ and a delta of 0.25 would result in $K_{25C-S-M}$ as the corresponding strike. Consequently, we would assign $\sigma_{ATM} + \sigma_{25-S-Q}$ to the strike $K_{25C-S-M}$ if we move from delta to the strike space. Similarly, a volatility of $\sigma_{ATM} + \sigma_{25-S-Q}$ would be assigned to $K_{25P-S-M}$. The resulting strangle from the three anchor smile would be

$$v(K_{25C-S-M}, \sigma_{ATM} + \sigma_{25-S-Q}, 1) + v(K_{25P-S-M}, \sigma_{ATM} + \sigma_{25-S-Q}, -1)$$

which is exactly the market strangle price v_{25-S-M}. In this particular case, we have

$$\begin{aligned} K_{25C-S-M} &= K_{25C}, \\ K_{25P-S-M} &= K_{25P}. \end{aligned}$$

Using the simplified smile construction procedure yields a market strangle consistent smile setup in case of a zero risk reversal. The other market matching requirements are met by default. In any other case, the strangle price might not be matched which leads to a non market consistent setup of the volatility smile.

The simplified formula can still be useful, even for large risk reversals, if σ_{25-S-Q}

is replaced by some other parameter introduced below. This parameter can be extracted after finishing the market consistent smile construction and is calculated in a way which is similar to Equation (30). Assume that the 0.25 delta volatilities $\sigma_{25C} = \overline{\sigma}(K_{25C})$ and $\sigma_{25P} = \overline{\sigma}(K_{25P})$ are given by the **calibrated** smile function $\overline{\sigma}(K)$. We can then calculate another strangle, called the smile strangle via

$$\sigma_{25-S-S} = \frac{\overline{\sigma}(K_{25C}) + \overline{\sigma}(K_{25P})}{2} - \sigma_{ATM}. \tag{31}$$

The smile strangle measures the convexity of the calibrated smile function and is plotted in Figure (4). It is approximately the difference between a straight line between the 25Δ put and call volatilities and the at-the-money volatility, evaluated at Δ_{ATM}. [3] This is equivalent to Equation (30), but in this case we are using out-

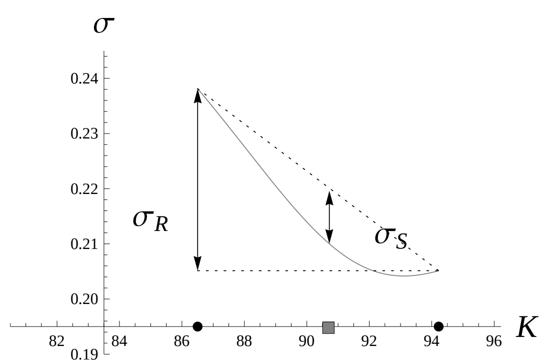

Fig. 4: Smile strangle for random market data. Filled circles indicate K_{25P}, K_{25C} strikes. Rectangle indicates K_{ATM}.

of-the-money volatilities obtained from the calibrated smile and not from the simplified formula. Given σ_{25-S-S}, the simplified Equation (28) can still be used if the quoted strangle volatility σ_{25-S-Q} is replaced by the smile strangle volatility σ_{25-S-S}. Clearly, σ_{25-S-S} is not known a priori but is obtained after finishing the calibration. Thus, one obtains a correct simplified formula as

$$\begin{aligned} \sigma_{25C} &= \sigma_{ATM} + \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-S}, \\ \sigma_{25P} &= \sigma_{ATM} - \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-S}. \end{aligned} \tag{32}$$

A sample data example is summarized in Table 8 where we have used the calibrated smile function $\overline{\sigma}(K)$ to calculate the smile strangles σ_{25-S-S}. Given σ_{25-S-S} , σ_{ATM} and σ_{25-RR}, we can calculate the EURUSD out-of-the-money volatilities of the call and put via the simplified Formula (32) as

[3] Here, Δ_{ATM} is the at-the-money delta. The description is exact if we consider the forward delta case with the delta-neutral at-the-money quotation. In other cases, this is an approximation.

Table 8: Smile strangle data

	EURUSD	USDJPY
$\overline{\sigma}(K_{25C})$	22.1092%	18.7693%
$\overline{\sigma}(K_{25P})$	22.6092%	24.0693%
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	−0.5%	−5.3%
σ_{25-S-S}	0.7377%	0.419%
σ_{25-S-Q}	0.7375%	0.184%

$$\sigma_{25C} = 21.6215\% - \frac{1}{2}0.5\% + 0.7377\% = 22.1092\%,$$
$$\sigma_{25P} = 21.6215\% + \frac{1}{2}0.5\% + 0.7377\% = 22.6092\%,$$

which is consistent with the volatilities $\overline{\sigma}(K_{25C})$ and $\overline{\sigma}(K_{25P})$ in Table 8. Note that the market strangle volatility is very close to the smile strangle volatility in the EURUSD case. This is due to the small risk reversal of the EURUSD smile. Calculating the 25Δ volatilities via the original simplified Formula (28) would yield a call volatility of 22.109% and a put volatility of 22.609% which are approximately the 0.25Δ volatilities of Table 8. However, the smile strangle and quoted strangle volatilities differ significantly for the skewed JPYUSD smile. Using the original Formula (28) in this case would result in 18.534% and 23.834% for the 25Δ call and put volatilities. These volatilities differ from the market consistent 25Δ volatilities given in Table 8.

Simplified Parabolic Interpolation

Various different interpolation methods can be considered as basic tools for the calibration procedure. Potential candidates are the SABR model introduced by Hagan et al. (2002), or the Vanna Volga method introduced by Castagna and Mercurio (2006). In this work, we introduce a new method for the smile construction. In a proceeding paper, we will compare all methods and analyze their calibration robustness empirically. The method introduced below turns out to be the most robust method.

In Malz (1997), the mapping forward delta against volatility is constructed as a polynomial of degree 2. This polynomial is constructed such that the at-the-money and risk reversal delta volatilities are matched. Malz derives the following functional relationship

$$\sigma(\Delta_f) = \sigma_{ATM} - 2\sigma_{25-RR}(\Delta_f - 0.5) + 16\sigma_{25-S-Q}(\Delta_f - 0.5)^2 \tag{33}$$

where Δ_f is a call forward delta[4]. This is a parabola centered at 0.5. The use of this functional relationship can be problematic due to the following set of problems:

- the interpolation is not a well defined volatility function since it is not always positive,
- the representation is only valid for forward deltas, although the author incorrectly uses the spot delta in his derivation (see Equation (7) and Equation (18) in Malz [1997]),
- the formula is only valid for the forward delta neutral at-the-money quotation,
- the formula is only valid for risk reversal and strangle quotes associated with a delta of 0.25,
- the matching of the market strangle restriction (25) is guaranteed for small risk reversals only.

The last point is crucial! If the risk reversal σ_{25-RR} is close to zero, the formula will yield $\sigma_{ATM} + \sigma_{25-S-Q}$ as the volatility for the ± 0.25 call and put delta. This is consistent with restriction (25). However, a significant risk reversal will lead to a failure of the formula. We will fix most of the problems by deriving a new, more generalized formula with a similar structure. The problem that the formula is restricted to a specific delta and at-the-money convention can be fixed easily. The matching of the market strangle will be employed by a suitable calibration procedure. The resulting equation will be denoted as the simplified parabolic formula.

The simplified parabolic formula is constructed in delta space. Let a general delta function $\Delta(K, \sigma, \phi)$ be given and K_{ATM} be the at-the-money strike associated with the given at-the-money volatility σ_{ATM}. Let the risk reversal volatility quote corresponding to a general delta of $\tilde{\Delta} > 0$ be given by $\sigma_{\tilde{\Delta}-RR}$. For the sake of a compact notation of the formula we will use σ_R instead of $\sigma_{\tilde{\Delta}-RR}$. Furthermore, we parametrize the smile by using a convexity parameter called **smile strangle** which is denoted as σ_S. This parameter has been discussed before in the simplified formula section. The following theorem can be stated.

Theorem 1. *Let Δ_{ATM} denote the call delta implied by the at-the-money strike*

$$\Delta_{ATM} = \Delta(K_{ATM}, \sigma_{ATM}, 1).$$

Furthermore, we define a variable $\boldsymbol{a}$ *which is the difference of a call delta, corresponding to a $-\tilde{\Delta}$ put delta, and the $-\tilde{\Delta}$ put delta for any delta type and is given by*

$$a := \Delta(K_{\tilde{\Delta}P}, \sigma, 1) - \Delta(K_{\tilde{\Delta}P}, \sigma, -1).$$

Given a call delta $\overline{\Delta}$, the parabolic mapping

$$(\overline{\Delta}, \sigma_S) \mapsto \sigma(\overline{\Delta}, \sigma_S)$$

which matches σ_{ATM} and the $\sigma_{\tilde{\Delta}-RR}$ risk reversal quote by default is

[4] A put volatility can be calculated by transforming the put to a call delta using the put call parity.

$$\sigma(\overline{\Delta}, \sigma_S) = \sigma_{ATM} + c_1(\overline{\Delta} - \Delta_{ATM}) + c_2(\overline{\Delta} - \Delta_{ATM})^2 \quad (34)$$

with

$$c_1 = \frac{a^2(2\sigma_S + \sigma_R) - 2a(2\sigma_S + \sigma_R)(\tilde{\Delta} + \Delta_{ATM}) + 2(\tilde{\Delta}^2\sigma_R + 4\sigma_S\tilde{\Delta}\Delta_{ATM} + \sigma_R\Delta_{ATM}^2)}{2(2\tilde{\Delta} - a)(\tilde{\Delta} - \Delta_{ATM})(\tilde{\Delta} - a + \Delta_{ATM})}$$

$$c_2 = \frac{4\tilde{\Delta}\sigma_S - a(2\sigma_S + \sigma_R) + 2\sigma_R\Delta_{ATM}}{2(2\tilde{\Delta} - a)(\tilde{\Delta} - \Delta_{ATM})(\tilde{\Delta} - a + \Delta_{ATM})} \quad (35)$$

assuming that the denominator of c_1 (and thus c_2) is not zero. A volatility for a put delta can be calculated via the transformation of the put delta to a call delta.

Proof: See Appendix.

We will present $\sigma(\Delta, \sigma_S)$ as a function depending on two parameters only, although of course more parameters are needed for the input. We consider σ_S explicitly, since this is the only parameter not observable in the market. This parameter will be the crucial object in the calibration procedure. Setting $\tilde{\Delta} = 0.25$, $\Delta_{ATM} = 0.5$ and $a = 1$ as in the forward delta case, yields the original Malz formula if $\sigma_S = \sigma_{25-S-Q}$. The generalized formula can handle any delta (e.g $\tilde{\Delta} = 0.10$), any delta type and any at-the-money convention. The formula automatically matches the at-the-money volatility, since

$$\sigma(\Delta_{ATM}, \sigma_S) = \sigma_{ATM}$$

Furthermore, the risk reversal is matched since

$$\sigma(\tilde{\Delta}_C, \sigma_S) - \sigma(a + \tilde{\Delta}_P, \sigma_S) = \sigma_{\tilde{\Delta}-RR}$$

where $\tilde{\Delta}_C$ denotes the call delta and $\tilde{\Delta}_P$ the put delta[5].

We have plotted the calibrated strike vs. volatility function in Figure (5) to show the influence of the parameters $\sigma_{ATM}, \sigma_R, \sigma_S$ on the simplified parabolic volatility smile in the strike space. We will explain later how to move from the delta to the strike space. Increasing σ_{ATM} leads to a parallel upper shift of the smile. Increasing σ_{25RR} yields to a more skewed curve. A risk reversal of zero implies a symmetric smile. Increasing the strangle σ_S increases the at-the-money smile convexity. Our final goal will be the adjustment of the smile convexity by changing σ_S until condition (25) is met. The other conditions are fulfilled by default, independent of the choice of σ_S.

We note that the simplified parabolic formula follows the sticky-delta rule. This implies, that the smile does not move in the delta space, if the spot changes (see Balland (2002), Daglish et al. (2007), Derman (1999)). In the strike space, the smile performs a move to the right in case of an increasing spot, see Figure (6) .

5 $a + \tilde{\Delta}_P$ is the call delta corresponding to a put delta of $\tilde{\Delta}_P$. In the forward delta case $a = 1$. If $\tilde{\Delta}_P = -0.25$, the equivalent call delta which enters the simplified parabolic formula is $a + \tilde{\Delta}_P = 0.75$.

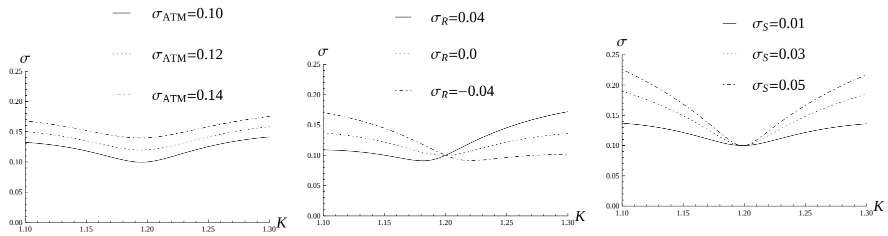

Fig. 5: Simplified Parabolic $\sigma_{ATM}, \sigma_R, \sigma_S$ spot delta scenarios with $\tau = \frac{35}{365}, S_0 = 1.2, r_d = 0.03, r_f = 0.01, \tilde{\Delta} = 0.25$. Initial parameters $\sigma_{ATM} = 10.0\%$, $\sigma_R = 0.6\%$, $\sigma_S = 1.0\%$.

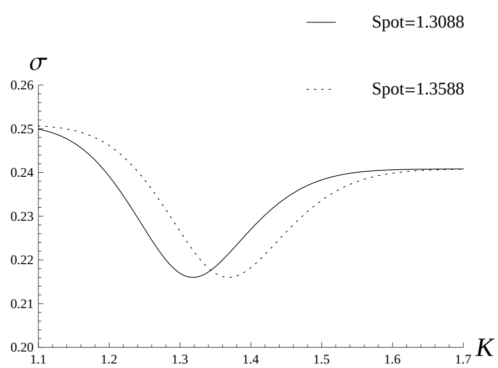

Fig. 6: Moving spot scenario for calibrated simplified parabolic formula in strike space. Based on market data in Table 3.

Market Calibration

The advantage of Formula (34) is that it matches the at-the-money and risk reversal conditions of Equations (17) and (18) by default. The only remaining challenge is matching the market strangle. The simplified parabolic function can be transformed from a delta-volatility to a strike-volatility space (which will be discussed later) such that a function

$$\sigma(K, \sigma_S)$$

is available. Using the variable σ_S as the free parameter, the calibration problem can be reduced to a search for a variable x such that the following holds

$$\begin{aligned} v_{\tilde{\Delta}-S-M} &= v(K_{\tilde{\Delta}C-S-M}, \sigma(K_{\tilde{\Delta}C-S-M}, x), 1) \\ &\quad + v(K_{\tilde{\Delta}P-S-M}, \sigma(K_{\tilde{\Delta}P-S-M}, x), -1). \end{aligned} \tag{36}$$

This leads to the following root search problem:

`Problem Type:`	Root search.
`Given parameters:`	$v_{\tilde{\Delta}-S-M}, K_{\tilde{\Delta}C-S-M}, K_{\tilde{\Delta}P-S-M}$ and market data.
`Target parameter:`	x (set x initially to $\sigma_{\tilde{\Delta}-S-Q}$)

`Objective function:`

$$f(x) = v(K_{\tilde{\Delta}C-S-M}, \sigma(K_{\tilde{\Delta}C-S-M}, x), 1) + v(K_{\tilde{\Delta}P-S-M}, \sigma(K_{\tilde{\Delta}P-S-M}, x), -1) - v_{\tilde{\Delta}-S-M}$$

The procedure will yield a smile strangle which can be used in the simplified parabolic formula to construct a full smile in the delta space. It is natural to ask, how well defined the problem above is and whether a solution exists. We will not present a rigorous analysis of this problem here, but it will be presented in follow-up research. We will show that a solution exists in a neighborhood of $\sigma_R = 0$ assuming that a weak condition is fulfilled. However, the neighborhood might be very small such that no solution for large risk-reversals might be available. The empirical tests in the following section will show, that the non-existence of such a solution has occurred in the past in very extreme market scenarios.
Performing the calibration on the currency data in Table 4 yields the parameters summarized in Table 7 for the root search problem. The final calibrated smile for

Table 7: Simplified Parabolic Calibration Results

	EURUSD Sample	USDJPY Sample
σ_S	0.007377	0.00419

the JPYUSD case is illustrated in Figure (8).

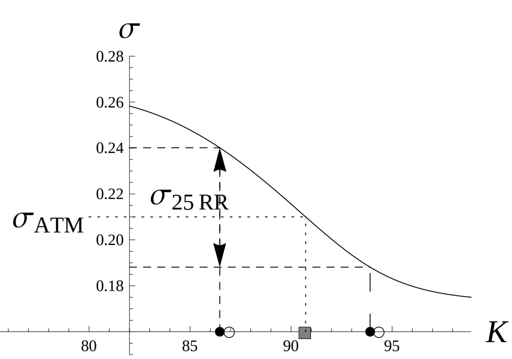

Fig. 8: JPYUSD smile for the market data in Exhibit 4. Filled circles indicate K_{25P}, K_{25C} strikes. Unfilled circles indicate market strangle strikes $K_{25P-S-M}, K_{25C-S-M}$. Rectangle indicates K_{ATM}.

Retrieving a Volatility for a Given Strike

Formula (34) returns the volatility for a given delta. However, the calibration procedure requires a mapping

$$K \mapsto \sigma(K, \sigma_S)$$

since it needs a volatility corresponding to the market strangle strikes. The transformation to $\sigma(K, \sigma_S)$ can be deduced by recalling that $\sigma = \sigma(\overline{\Delta}, \sigma_S)$ is the volatility corresponding to the delta $\overline{\Delta}$. To be more precise, given that σ is assigned to delta $\overline{\Delta}$ implies that $\overline{\Delta} = \Delta(K, \sigma, \phi)$ for some strike K. Consequently, Formula (34) can be stated as

$$\sigma = \sigma_{ATM} + c_1(\Delta(K, \sigma, 1) - \Delta_{ATM}) + c_2(\Delta(K, \sigma, 1) - \Delta_{ATM})^2. \tag{37}$$

Given a strike K, it is thus possible to retrieve the corresponding volatility by searching for a σ which fulfills Equation (37). This can be achieved by using a root searcher. We recommend the method introduced by Brent (2002). The question arises, if such a volatility vs. strike function exists and how smooth it is. The answer can be given by using the implicit function theorem. In the following discussion we will avoid the explicit dependence of all variables on $(K, \sigma(K, \sigma_S))$. For example, we write

$$\frac{\partial \Delta}{\partial K} \text{ instead of } \frac{\partial \Delta}{\partial K}(x,y)|_{x=K, y=\sigma(K,\sigma_S)}$$

With this compact notation, we can state the following.

Theorem 2. *Given the volatility vs. delta mapping (34), assume that the following holds*

$$c_1 \frac{\partial \Delta}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) \neq 1$$

*Then there exists a function $\sigma : U \to W$ with open sets $U, W \subseteq \mathbb{R}^+$ such that $K_{ATM} \in U$ and $\sigma_{ATM} \in W$ which maps the strike implicit in Δ against the corresponding volatility. The function is **differentiable** and has the following first- and second-order derivatives on U*

$$\frac{\partial \sigma}{\partial K} = \frac{\frac{\partial \Delta}{\partial K} A}{1 - \frac{\partial \Delta}{\partial \sigma} A} \tag{38}$$

$$\begin{aligned} \frac{\partial^2 \sigma}{\partial K^2} &= \frac{\left[\left(\frac{\partial^2 \Delta}{\partial K^2} + \frac{\partial^2 \Delta}{\partial K \partial \sigma}\frac{\partial \sigma}{\partial K}\right)A + \frac{\partial \Delta}{\partial K}\frac{\partial A}{\partial K}\right]\left(1 - \frac{\partial \Delta}{\partial \sigma}A\right)}{\left(1 - \frac{\partial \Delta}{\partial \sigma}A\right)^2} \\ &+ \frac{\frac{\partial \Delta}{\partial K}A\left((\frac{\partial \Delta}{\partial \sigma \partial K} + \frac{\partial^2 \Delta}{\partial \sigma^2}\frac{\partial \sigma}{\partial K})A + \frac{\partial \Delta}{\partial \sigma}\frac{\partial A}{\partial K}\right)}{\left(1 - \frac{\partial \Delta}{\partial \sigma}A\right)^2} \end{aligned} \tag{39}$$

with

$$A := c_1 + 2c_2(\Delta - \Delta_{ATM}) \text{ and } \frac{\partial A}{\partial K} = 2c_2\left(\frac{\partial \Delta}{\partial K} + \frac{\partial \Delta}{\partial \sigma}\frac{\partial \sigma}{\partial K}\right)$$

Proof. See Appendix.

Note that Equations (38) and (39) require the values $\sigma(K, \sigma_S)$. In fact, Equation (38) can be seen as an non-autonomous non-linear ordinary differential equation for $\sigma(K, \sigma_S)$. However, given $\sigma(K, \sigma_S)$ as a root of Equation (37), we can analytically calculate both derivatives. Differentiability is very important for calibration procedures of the well known local volatility models (see Dupire (1994), Derman and Kani (1994), Lee (2001)), which need a smooth volatility vs. strike function. To be more precise, given the local volatility SDE

$$dS_t = (r_d - r_f)S_t dt + \sigma(S_t, t)dW_t$$

the function $\sigma(K,t)$ can be stated in terms of the implied volatility (see Andersen and Brotherton-Ratcliffe (1998), Dempster and Richards (2000)) as

$$\sigma^2(K,T) = \frac{2\frac{\partial \sigma}{\partial T} + \frac{\sigma}{T-t} + 2K(r_d - r_f)\frac{\partial \sigma}{\partial K}}{K^2\left[\frac{\partial^2 \sigma}{\partial K^2} - d_+\sqrt{T-t}(\frac{\partial \sigma}{\partial K})^2 + \frac{1}{\sigma}\left(\frac{1}{K\sqrt{T-t}} + d_+\frac{\partial \sigma}{\partial K}\right)^2\right]}.$$

The derivatives with respect to the strike can be very problematic if calculated numerically from an interpolation function. In our case, the derivatives can be stated explicitly, similar to (Hakala and Wystup, 2002, page 254) for the kernel interpolation case. In addition, the formulas are very useful to test for arbitrage, where restrictions on the slope and convexity of $\sigma(K)$ are imposed (see for example Lee (2005)).

We summarize explicit formulas for all derivatives occurring in Equations (38) and (39) in Tables 10 and 11 in the Appendix. They can be used for derivations of analytical formulas for the strike derivatives for all delta types.

Extreme Strike Behavior

Lee (2004) published a very general result about the extreme strike behavior of any implied volatility function. Work in this area has been continued by Benaim, Friz and Lee in Benaim et al. (2009), Benaim and Friz (2009). The basic idea of Lee is the following. Let

$$x := \ln\left(\frac{K}{f}\right)$$

be the log-moneyness and $I^2(x)$ the implied variance for a given moneyness x. Independent of the underlying model for the asset S there exists a $\beta_R \in [0,2]$ such that

$$\beta_R := \limsup_{x\to\infty} \frac{I^2(x)}{|x|/T}.$$

A very important result is that the number β_R is directly related to the highest finite moment of the underlying S at time T such that β_R can be stated more explicitly depending on the model. Define

$$\tilde{p} := \sup\{p : E(S_T^{1+p}) < \infty.\}$$

then we have

$$\beta_R = 2 - 4(\sqrt{\tilde{p}^2 + \tilde{p}} - \tilde{p}),$$

where the right hand expression is to be read as zero in the case $\tilde{p} = \infty$. A similar expression can be obtained for $x \to -\infty$. Consequently, the modeling of the implied volatility function in the delta space can not be arbitrarily, since Lee's extreme strike behavior has to be fulfilled. In the Appendix, we prove the following extreme strike behavior for the simplified parabolic formula:

$$\lim_{x\to\infty} \sigma(\Delta_S(x), \sigma_S) = \sigma_{ATM} - c_1\Delta_{ATM} + c_2\Delta_{ATM}^2, \tag{40}$$

which is a constant. Similarly,

$$\lim_{x\to-\infty} \sigma(\Delta_S(x), \sigma_S) = \sigma_{ATM} + c_1(e^{-r_f\tau} - \Delta_{ATM}) + c_2(e^{-r_f\tau} - \Delta_{ATM})^2, \tag{41}$$

which is again a constant. Equivalent results can be derived for the forward delta and the premium-adjusted versions. Consequently, the simplified formula implies a constant extrapolation, which is consistent with Lee's moment formula. The constant extrapolation implies that

$$\lim_{x\to\infty} \frac{I(x)}{\sqrt{|x|/T}} = 0 = \lim_{x\to-\infty} \frac{I(x)}{\sqrt{|x|/T}}.$$

This is only consistent, if

$$\sup\{p : E(S_T^{p+1}) < \infty\} = \infty,$$

e.g. all moments of the underlying at time T are finite. Although the simplified parabolic formula has been derived with a rather heuristic argumentation, it is only consistent if the underlying that generates such a volatility smile has finite moments of all orders.

Potential Problems

Potential numerical issues may arise due to the following:

1. Formula (34) is not restricted to yield positive values.

2. A root for Equation (37) might not exist. We do not know how large U, W are and whether a volatility can be found for any strike K.
3. The denominator in equation system (35) can be zero.
4. A root for Equation (36) might not exist.

The question arises, how often these problems occur in the daily market calibration. We have analyzed the occurrence of the problems above based on market data published on Bloomberg, where $\sigma_{ATM}, \sigma_{10-RR}, \sigma_{25-RR}$ and $\sigma_{10-S-Q}, \sigma_{25-S-Q}$ volatilities are quoted. We have considered the currencies EUR, GBP, JPY, CHF, CAD and AUD, which account for 88% of the worldwide traded OTC derivative notionals[6]. The data is summarized in Figure (9). The volatilities are quoted for maturities of $1, 3, 6, 9$ and 12 months. The delta types for all maturities below 9 months are spot deltas for the currency pairs EURUSD, GBPUSD, AUDUSD and premium-adjusted spot deltas for the currency pairs USDJPY, USDCHF, USDCAD. For the 12 month maturity, the first currency group uses forward deltas, while the second one uses premium-adjusted forward deltas. All currencies use the forward delta neutral straddle as the at-the-money convention. We have performed a daily calibration

Table 9: FX Data Summary

	EURUSD	GBPUSD	USDJPY	USDCHF	USDCAD	AUDUSD
Begin Date	03.10.2003	03.10.2003	03.10.2003	05.01.2006	03.10.2003	03.10.2003
End Date	20.01.2009	20.01.2009	20.01.2009	20.01.2009	20.01.2009	20.01.2009
Data Sets	5834	5780	6126	3849	5775	5961

to market data for all maturities and currencies. The calibrations were performed to the 0.25Δ and 0.10Δ quotes separately. Then we have tested for problems occurring within a $\pm 0.10\Delta$ range. A check for a zero denominator in equation system (35) has been performed. Finally, we checked the existence of a root for the implied problem (37). In **none** of the more than $30,000$ calibrations did we observe any of the first three problems. We thus conclude, that the method is very robust in the daily calibration.

However, the calibration failed 6 times (in more than $30,000$ calibrations) in the root searching procedure for Equation (36). This happened for the 0.10Δ case for the extremely skewed currency pair JPYUSD, where risk reversals of 19% and more were observed in the extreme market scenarios following the financial crisis. The calibration procedure is more robust than other methods which have shown more than 300 failures in some cases. Also, it is not obvious whether any smile function can match the market quotes in these extreme scenarios. These issues will be covered in future research.

[6] Based on data as of December 2008, published by the Bank for International Settlements on www.bis.org/publ/qtrpdf/r_qa0906.pdf

2 Conclusion

We have introduced various delta and at-the-money quotations commonly used in FX option markets. The delta types are FX-specific, since the option can be traded in both currencies. The various at-the-money quotations have been designed to account for large interest rate differentials or to enforce an efficient trading of positions with a pure vega exposure. We have then introduced the liquid market instruments that parametrize the market and have shown which information they imply. Finally, we derived a new formula that accounts for FX specific market information and can be used to employ an efficient market calibration.

Follow-up research will compare the robustness and potential problems of different smile calibration procedures by using empirical data. Also, potential calibration problems in extreme market scenarios will be analyzed.

Acknowledgments

We would like to thank Travis Fisher, Boris Borowski, Andreas Weber, Jürgen Hakala and Ian Clark for their helpful comments.

3 Appendix

To reduce the notation, we will drop the dependence of $\sigma(\Delta, \sigma_S)$ on σ_S in the following proofs and write $\sigma(\Delta)$ instead.

Proof (Simplified Parabolic Formula). We will construct a parabola in the call delta space such that the following restrictions are met

$$\begin{aligned} \sigma(\Delta_{ATM}) &= \sigma_{ATM}, \\ \sigma(\tilde{\Delta}) &= \sigma_{ATM} + \frac{1}{2}\sigma_R + \sigma_S, \\ \sigma(a - \tilde{\Delta}) &= \sigma_{ATM} - \frac{1}{2}\sigma_R + \sigma_S. \end{aligned} \tag{42}$$

For example, in the forward delta case we would have $a = 1$. Given $\tilde{\Delta} = 0.25$, the call delta corresponding to a put delta of -0.25 would be $1 - 0.25 = 0.75$. The equation system is set up such that

$$\sigma_S = \frac{\sigma(\tilde{\Delta}) + \sigma(a - \tilde{\Delta})}{2} - \sigma_{ATM}.$$

One can see that σ_S measures the smile convexity, as it is the difference of the average of the out-of-the-money and in-the-money volatilities compared to the at-the-money volatility. The restriction set (42) ensures that

$$\sigma(\tilde{\Delta}) - \sigma(a - \tilde{\Delta}) = \sigma_R \tag{43}$$

is fulfilled by default. Given the parabolic setup

$$\sigma(\overline{\Delta}) = \sigma_{ATM} + c_1(\overline{\Delta} - \Delta_{ATM}) + c_2(\overline{\Delta} - \Delta_{ATM})^2,$$

one can solve for c_1, c_2 such that Equation system (42) is fulfilled. This is a well defined problem: a system of two linear equations in two unknowns. $\square$

Proof (Existence of a Volatility vs Strike Function). The simplified parabolic function has the following form

$$\sigma(\Delta, \sigma_S) = \sigma_{ATM} + c_1(\Delta - \Delta_{ATM}) + c_2(\Delta - \Delta_{ATM})^2. \tag{44}$$

First of all, note that $\Delta(K, \sigma)$ is continuously differentiable with respect to both variables for all delta types. Define $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ to be

$$F(K, \sigma) = \sigma_{ATM} + c_1(\Delta(K, \sigma) - \Delta_{ATM}) + c_2(\Delta(K, \sigma) - \Delta_{ATM})^2 - \sigma \tag{45}$$

with $\Delta(K, \sigma)$ being one of the four deltas introduced before. The proof is a straightforward application of the implicit function theorem. Note that $F(K_{ATM}, \sigma_{ATM}) = 0$ is given by default. As already stated, the function F is differentiable with respect to the strike and volatility. Deriving with respect to volatility yields

$$\frac{\partial F}{\partial \sigma} = c_1 \frac{\partial \Delta}{\partial \sigma} + 2c_2(\Delta - \Delta_{ATM}) \frac{\partial \Delta}{\partial \sigma} - 1. \tag{46}$$

From this derivation we have

$$\frac{\partial F}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) = c_1 \frac{\partial \Delta}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) - 1, \tag{47}$$

which is different from zero by assumption of the theorem. Consequently, the implicit function theorem implies the existence of a differentiable function f and an open neighborhood $U \times W \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ with $K_{ATM} \in U$, $\sigma_{ATM} \in W$ such that

$$F(K, \sigma) = 0 \Leftrightarrow \sigma = f(K) \text{ for } (K, \sigma) \in U \times W.$$

The first derivative is defined on U and given by

$$\frac{\partial f}{\partial K} = -\frac{\frac{\partial F}{\partial K}}{\frac{\partial F}{\partial \sigma}} \text{ for } K \in U,$$

which can be calculated in a straightforward way. The function $f(K)$ is denoted as $\sigma(K)$ in the theorem. The second derivative can be derived in a straightforward way by remembering, that the volatility depends on the strike. This completes the proof. □

Proof (Extreme Strike Behavior of Simplified Parabolic Interpolation). Let

$$x := \log\left(\frac{K}{f}\right)$$

be the log moneyness. The terms $d_\pm$ can be rewritten as

$$d_\pm(x) := \frac{-x \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

We then have:

$$\lim_{x\to\infty} N(d_\pm(x)) = 0, \tag{48}$$

$$\lim_{x\to-\infty} N(d_\pm(x)) = 1. \tag{49}$$

The c_1, c_2 terms are constants. Consequently, for the spot delta we derive:

$$\lim_{x\to\infty} \sigma(\Delta(x), \sigma_S) = \sigma_{ATM} - c_1\Delta_{ATM} + c_2\Delta_{ATM}^2, \tag{50}$$

which is a constant. Similarly,

$$\lim_{x\to-\infty} \sigma(\Delta(x), \sigma_S) = \sigma_{ATM} + c_1(e^{-r_f\tau} - \Delta_{ATM}) + c_2(e^{-r_f\tau} - \Delta_{ATM})^2, \tag{51}$$

which is again a constant. Equivalent results can be derived for the forward delta. The next analysis discusses the premium adjusted forward delta case; the spot premium adjusted case is similar. Rewriting the premium adjusted forward delta in terms of the log moneyness x yields

$$\Delta_{fpa} = e^x N(d_-(x)) = e^x N\left(-\left[\frac{x + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right]\right) = e^x - e^x N\left(\frac{x + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right).$$

Consequently, we have

$$\lim_{x\to\infty} \Delta_{f,pa}(x) = 0 = \lim_{x\to-\infty} \Delta_{f,pa}(x).$$

This implies that

$$\lim_{x\to\infty} \sigma(\Delta_{f,pa}(x), \sigma_S) = \sigma_{ATM} - c_1\Delta_{ATM} + c_2\Delta_{ATM}^2 = \lim_{x\to-\infty} \sigma(\Delta_{f,pa}(x), \sigma_S). \tag{52}$$

Note, that this limit differs from the spot delta case, since the terms a and Δ_{ATM} are different. □

	∂K	$\partial \sigma$	∂K^2
Δ_S	$-\frac{e^{-r_f\tau}n(d_+)}{\sigma\sqrt{\tau}K}$	$-\frac{e^{-r_f\tau}n(d_+)d_-}{\sigma}$	$\frac{e^{-r_f\tau}n(d_+)}{\sigma\sqrt{\tau}K^2} - \frac{e^{-r_f\tau}n(d_+)d_+}{\sigma^2\tau K^2}$
$\Delta_{S,pa}$	$\frac{\phi e^{-r_f\tau}N(\phi d_-)}{f} - \frac{e^{-r_f\tau}n(d_-)}{f\sigma\sqrt{\tau}}$	$-\frac{e^{-r_f\tau}Kn(d_-)d_+}{f\sigma}$	$-\frac{e^{-r_f\tau}n(d_-)}{f\sigma\sqrt{\tau}K} - \frac{e^{-r_f\tau}n(d_-)d_-}{fK\sigma^2\tau}$
Δ_f	$-\frac{n(d_+)}{\sigma\sqrt{\tau}K}$	$-\frac{n(d_+)d_-}{\sigma}$	$\frac{n(d_+)}{\sigma\sqrt{\tau}K^2} - \frac{n(d_+)d_+}{\sigma^2\tau K^2}$
$\Delta_{f,pa}$	$\frac{\phi N(\phi d_-)}{f} - \frac{n(d_-)}{f\sigma\sqrt{\tau}}$	$-\frac{Kn(d_-)d_+}{f\sigma}$	$-\frac{n(d_-)}{f\sigma\sqrt{\tau}K} - \frac{n(d_-)d_-}{fK\sigma^2\tau}$

Table 10: Partial Delta Derivatives I

	$\partial K \partial \sigma$	$\partial \sigma^2$
Δ_S	$\frac{e^{-r_f\tau}n(d_+)\left(1-d_+d_-\right)}{\sigma^2\sqrt{\tau}K}$	$\frac{e^{-r_f\tau}n(d_+)(d_- - d_+d_-d_- + d_+)}{\sigma^2}$
$\Delta_{S,pa}$	$\frac{e^{-r_f\tau}n(d_-)\left(-d_+\sigma\sqrt{\tau}+1-d_-d_+\right)}{f\sigma^2\sqrt{\tau}}$	$\frac{e^{-r_f\tau}Kn(d_-)(d_+ - d_-d_+d_+ + d_-)}{f\sigma^2}$
Δ_f	$\frac{n(d_+)\left(1-d_+d_-\right)}{\sigma^2\sqrt{\tau}K}$	$\frac{n(d_+)(d_- - d_+d_-d_- + d_+)}{\sigma^2}$
$\Delta_{f,pa}$	$\frac{n(d_-)\left(-d_+\sigma\sqrt{\tau}+1-d_-d_+\right)}{f\sigma^2\sqrt{\tau}}$	$\frac{Kn(d_-)(d_+ - d_-d_+d_+ + d_-)}{f\sigma^2}$

Table 11: Partial Delta Derivatives II

References

Andersen, L. and R. Brotherton-Ratcliffe, "The equity option volatility smile: an implicit finite-difference approach," *Journal of Computational Finance*, 1998, *1* (2), 5–37.

Balland, P., "Deterministic implied volatility models," *Quantitative Finance*, 2002, *2* (1), 31–44.

Beier, C.C. and C. Renner, "Foreign Exchange Options: Delta- and At-the-money Conventions," in "Encyclopedia of Quantitative Finance (Rama Cont and Peter

Tankov Eds.),” Wiley, 2010.

Benaim, S. and P. Friz, “Regular variation and smile asymptotics,” *Mathematical Finance*, 2009, *19* (1), 1–12.

— , — , **and R. Lee**, “The Black-Scholes Implied Volatility at Extreme Strikes,” *Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling*, 2009, pp. 18–46.

Beneder, R. and M. Elkenbracht-Huizing, “Foreign Exchange Options and the Volatility Smile,” *Medium Econometrische Toepassingen*, 2003, *2*, 30–36.

Bossens, F., G. Rayée, N.S. Skantzos, and G. Deelstra, “Vanna-Volga methods applied to FX derivatives: from theory to market practice,” Working Papers CEB, Université Libre de Bruxelles, Solvay Brussels School of Economics and Management, Centre Emile Bernheim (CEB) Apr 2009.

Brent, R.P., *Algorithms for minimization without derivatives*, Courier Dover Publications, 2002.

Castagna, A., *FX Options and Smile Risk*, Wiley, 2010.

— **and F. Mercurio**, “Consistent Pricing of FX Options,” *SSRN eLibrary*, 2006.

Clark, I.J., *Foreign Exchange Option Pricing: A Practitioner's Guide*, Wiley, forthcoming.

Daglish, T., J. Hull, and W. Suo, “Volatility surfaces: theory, rules of thumb, and empirical evidence,” *Quantitative Finance*, 2007, *7* (5), 507–524.

Dempster, M.A.H. and D.G. Richards, “Pricing American options fitting the smile,” *Mathematical Finance*, 2000, *10* (2), 157–177.

Derman, E., “Regimes of volatility,” *Risk*, 1999, *4*, 55–59.

— **and I. Kani**, “Riding on a smile,” *Risk*, 1994, *7* (2), 32–39.

Dupire, B., “Pricing with a smile,” *Risk*, 1994, *7* (1), 18–20.

Hagan, P.S., D. Kumar, A.S. Lesniewski, and D.E. Woodward, “Managing smile risk,” *Wilmott Magazine*, 2002, *1*, 84–108.

Hakala, J. and U. Wystup, *Foreign Exchange Risk*, Risk Books, 2002.

Lee, R.W., “Implied and local volatilities under stochastic volatility,” *International Journal of Theoretical and Applied Finance*, 2001, *4* (1), 45–89.

— , “The moment formula for implied volatility at extreme strikes,” *Mathematical Finance*, 2004, *14* (3), 469–480.

— , “Implied volatility: Statics, dynamics, and probabilistic interpretation,” *Recent Advances in Applied Probability*, 2005, pp. 241–268.

Malz, A.M., “Option-Implied Probability Distributions and Currency Excess Returns,” *SSRN eLibrary*, 1997.

Wystup, U., *FX Options and Structured Products*, Wiley, 2006.